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Eigenoscillations of a fluid in a canonical domain and functional difference equations

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HIGHLIGHTS

- Explicit formulas for the eigenmodes of the continuous spectrum are considered.
- New type of the functional-difference equations are solved.
- The far field asymptotics of the eigenfunctions are obtained.
- Some physical analysis of the solution for the problem at hand is also given.

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ABSTRACT

In this work we construct and discuss special solutions of a homogeneous problem for the Laplace equation in a domain with cone-shaped boundaries. The problem at hand is interpreted as that describing oscillatory linear wave movement of a fluid under gravity in such a domain. These solutions are found in terms of the Mellin transform and by means of the reduction to some new functional-difference equations solved in an explicit form (by quadrature). The behavior of the solutions at large distances is studied by use of the saddle point technique. The corresponding eigenoscillations of a fluid are then interpreted as generalized eigenfunctions of the continuous spectrum.

1. Introduction

Let us consider a linear time-harmonic water-wave process described by the velocity potential $U(X, Y, Z, t)$; see e.g. [1]. The complex velocity potential u is connected with the unsteady potential

$$U(X, Y, Z, t) = \Re\{\exp(-\mathrm{i}\Omega t)u(X, Y, Z)\},$$

where Ω is the angular frequency [1].

The velocity potential satisfies the Laplace equation

$$\triangle U(X, Y, Z, t) = 0$$

in the domain W, see Fig. 1, $\triangle = \partial_X^2 + \partial_Y^2 + \partial_Z^2$, $t > 0$, the dynamic boundary condition on the free surface F $(Z = 0,\ |X| > 0,\ |Y| > 0)$

$$U_{tt} - g\,U_Z = 0, \quad t > 0,$$

where g is the gravitational acceleration.

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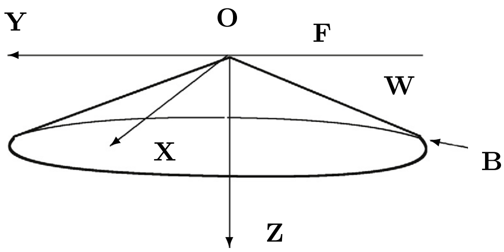

Fig. 1. Canonical domain.

The boundary condition on the conical surface of the bottom B can be taken in the form

$$U_n + \eta^{-1} U_t = 0, \quad t > 0,$$

where U_n is the derivative of the potential with respect to the normal to B directed into W. The boundary condition on the bottom requires a comment. The summand $\eta^{-1} U_t$ is responsible for the process of infiltration of fluid through the bottom, the parameter η specifies the 'velocity' of the infiltration. If the value η^{-1} turns out to be small and the effect of the infiltration is negligible, one can use the model of rigid bottom $U_n|_B = 0$. In the model of rigid bottom this term $\eta^{-1} U_t$ is absent so that formally we put $\eta^{-1} = 0$. In this work, we consider the opposite limiting case $\eta^{-1} = \infty$ of 'perfect' infiltration and assume that $U_t|_B = 0$ which is valid provided that the Dirichlet boundary condition

$$u(X, Y, Z)|_B = 0$$

is satisfied on the bottom for the stationary problem.[1] In addition, the potential is such that $U(X, Y, 0, 0)$ and $U_t(X, Y, 0, 0)$ i.e are assumed to be known at the initial moment of time $t = 0$. We may assume that the free surface is at equilibrium at the initial moment.

The domain shown in Fig. 1 might serve to simulate water-wave processes near a shallow or sandbank in the ocean. For this reason such a canonical problem may be called the linear water-wave problem for a sandbank. At the same time one can encounter similar geometry when studying the water-wave processes near a small island or atoll in the ocean.

The stationary complex potential $u(X, Y, Z)$ is governed by the Laplace equation, see the next section for the complete formulation. The boundary condition with the spectral parameter is valid on the free water surface F, whereas on the bottom B of the conical shape the Dirichlet condition is satisfied. The boundary condition on the free water surface F arises from the nonstationary one discussed above. Recall that the problem is considered in the linear small-amplitude wave approximation. By means of the Mellin integral representation we separate the radial variable. The solution is reduced to that for a problem in a domain of the unit sphere. Some new functional-difference equations are then derived and solved, which leads to a closed form representation for the corresponding eigenfunctions. By use of the saddle point technique the Mellin integral is then asymptotically evaluated at large distances from the vertex. The obtained classical solutions of the homogeneous problem, i.e. eigenoscillations, are interpreted as eigenfunctions of the continuous spectrum of infinite multiplicity.

It should be mentioned, however, that different kinds of functional equations were considered in fluid mechanics [2,3], in diffraction theory [4–12], in theoretical and mathematical physics [13,14]. It is worth remarking that the study and solution of some new functional equations is at the core of the present research. Contrary to the trigonometric coefficients for the Malyuzhinets' equations, this new type of the equations has more complex coefficients which depend on the associated Legendre functions. The equations considered in the present work are similar (but not the same) to those studied in our work [3]. Additionally, in comparison with that in [3] we give detailed the asymptotic analysis of the new eigensolutions based on the saddle point technique and discuss their physical interpretation.

2. Formulation of the problem

The Cartesian coordinates X, Y, Z are connected with the spherical ones in W by the formulas

$$X = r\cos\varphi \sin\theta, \quad Y = r\sin\varphi\sin\theta, \quad Z = r\cos\theta,$$

the axis OZ directed vertically downwards as shown in Fig. 1. Let the fluid medium occupy the conical domain, $\omega = (\theta, \varphi)$

$$W = \left\{(r, \omega): \quad r > 0, \quad -\pi < \varphi \le \pi, \quad \theta_1 < \theta < \frac{\pi}{2}\right\},$$

and $\theta = \theta_1$ is the equation of the bottom B, $0 < \theta_1 < \frac{\pi}{2}$.

[1] Although, in a sense, such classical condition might be considered artificial in the theory of linear gravity water waves, the corresponding problem is of interest from the mathematical point of view.

The desired potential satisfies the Laplace equation

$$\triangle u(r,\omega)=0 \qquad \text{in} \quad W, \tag{1}$$

$$\triangle=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\triangle_\omega,\quad \triangle_\omega=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$

the boundary condition on the free surface F ($\theta=\frac{\pi}{2}$)

$$\left.\left(\frac{1}{r}\frac{\partial}{\partial\theta}u(r,\theta,\varphi)-K\,u(r,\theta,\varphi)\right)\right|_{\theta=\frac{\pi}{2}}=0 \qquad \text{on} \quad F, \tag{2}$$

with the spectral parameter $K=\frac{\Omega^2}{g}$ and

$$u(r,\theta,\varphi)|_{\theta=\theta_1}=0 \qquad \text{on} \quad B. \tag{3}$$

We are looking for the classical solution of (1)–(3) which is $2\pi-$periodic in φ,

$$u(r,\theta,\varphi)=u(r,\theta,\varphi+2\pi). \tag{4}$$

The Meixner type conditions at the vertex O of the conical domain W are written in the form

$$|u(r,\theta,\varphi)|\le \mathrm{const}\, r^{\delta}, \qquad r|\nabla u(r,\theta,\varphi)|\le \mathrm{const}\, r^{\delta}, \tag{5}$$

which is valid uniformly with respect to the angular variables, for some $\delta>-\frac{1}{2}$.

Introduce the functional of energy (see also [1] for the appropriate terminology)

$$E(u)=\int_W |\nabla u|^2\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z\;+\;K\int_F |u|^2\,\mathrm{d}X\mathrm{d}Y, \tag{6}$$

where the first and second integrals in (6) are usually associated with the kinetic and potential energy. It is worth noting that the spectral parameter K is in the boundary condition, whereas its presence in the equation is more traditional for spectral analysis of the boundary-value problems. For this kind of problems, however, we accept a standard terminology and call a non-trivial solution u of (1)–(5) an eigenfunction of the point spectrum with eigenvalue K provided

$$E(u)\;<\;\infty\,. \tag{7}$$

In view of the definition (6) the condition (7) implies that u and $|\nabla u|$ decay at infinity rapidly enough so that the estimate

$$|u(r,\theta,\varphi)|\le \mathrm{const}\, r^{-\varkappa}, \quad r\;\to\;\infty \tag{8}$$

is valid uniformly w.r.t. angular variables $\omega:=(\theta,\varphi)$ for some $\varkappa>1$.

However, for some given K a solution u of the homogeneous problem may decay slower than that in (8) or even grow at most polynomially so that the integrals in (6) diverge. We call such solutions the generalized eigenfunctions of the *continuous* spectrum or, for simplicity, eigenfunctions of the continuous spectrum . We consider $K>0$ assuming that $K=0$ is non-physical. It is remarkable that the corresponding solutions can be determined in an explicit form.

3. Mellin integral representation and reduction to the functional equations

We separate the radial variable and determine the solution in the form (see e.g. [15]) of the Mellin integral

$$u(r,\omega)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} u_\nu(\omega)\,r^{\nu-\frac{1}{2}}\,\mathrm{d}\nu, \tag{9}$$

where c is a constant, $\omega=(\theta,\varphi)$. The inverse is

$$u_\nu(\omega)=\int_0^\infty u(r,\omega)\,r^{-\nu-\frac{1}{2}}\,\mathrm{d}r. \tag{10}$$

The representation (9) satisfies Meixner's conditions, e.g. $u(r,\theta,\varphi)=O(r^a)$ as $r\to 0$, and vanishes at infinity, i.e. $u(r,\theta,\varphi)=O(r^{-b})$ as $r\to\infty$, we conclude that the Mellin transformant u_ν in (10) is regular in the strip

$$\frac{1}{2}-b<\Re(\nu)<\frac{1}{2}+a, \tag{11}$$

where it is assumed that $a>-b$ with $a>c-\frac{1}{2}>-b$.

The function $u(\cdot,\omega)$ is locally integrable on $[0,\infty)$. The unknown u_ν is sought in the class of meromorphic functions of ν for any $\omega\in\Sigma$, where Σ is the spherical layer on the unit sphere S^2 with center at the origin O such that

$$\Sigma=\{\omega:\quad \omega\in S^2\cap W\}.$$

The boundary of Σ consists of two circles on S^2 with the equations $\theta = \theta_1$ and $\theta = \frac{\pi}{2}$. We also require that

$$u_\nu(\omega) \to 0, \quad |\nu| \to \infty$$

as ν belongs to the strip (11) for $\omega \in \overline{\Sigma}$.

We easily verify that, provided the transformed function $u_\nu(\cdot)$ satisfies the equation

$$\left(\triangle_\omega + \nu^2 - \frac{1}{4}\right) u_\nu(\omega) = 0 \quad \text{on } \Sigma, \tag{12}$$

the original u solves the Laplace equation (1) in W, [3].

We find the boundary conditions for $u_\nu(\cdot)$. Substituting (9) into the boundary condition (2) on F, one has

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left(\frac{\partial u_\nu(\omega)}{\partial \theta} r^{\nu-\frac{3}{2}} - K u_\nu(\omega) r^{\nu-\frac{1}{2}} \right) \mathrm{d}\nu \bigg|_{\theta=\frac{\pi}{2}} =$$

$$\left\{ \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty-1}^{+\mathrm{i}\infty-1} \frac{\partial u_{\nu+1}(\omega)}{\partial \theta} r^{\nu-\frac{1}{2}} \,\mathrm{d}\nu - \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} K u_\nu(\omega) r^{\nu-\frac{1}{2}} \,\mathrm{d}\nu \right\} \bigg|_{\theta=\frac{\pi}{2}} = 0,$$

where we used the change of the integration variable $\nu - 1 \to \nu$ in the first integral of the last line. Then we deform the straight-line contour $(-\mathrm{i}\infty - 1, +\mathrm{i}\infty - 1)$ in this integral back onto the imaginary axis assuming that singularities of $\frac{\partial u_{\nu+1}(\omega)}{\partial \theta}\Big|_{\theta=\frac{\pi}{2}}$ are not crossed in this process, which can be justified *a posteriori*. We find

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left(\frac{\partial u_{\nu+1}(\omega)}{\partial \theta} - K u_\nu(\omega) \right) \bigg|_{\theta=\frac{\pi}{2}} r^{\nu-\frac{1}{2}} \,\mathrm{d}\nu = 0,$$

which is valid provided the condition

$$\left(\frac{\partial u_{\nu+1}(\omega)}{\partial \theta} - K u_\nu(\omega) \right) \bigg|_{\theta=\frac{\pi}{2}} = 0 \tag{13}$$

is satisfied.

It is worth noticing that the boundary condition (13) is non-local with respect to ν. We separated the radial variable in the mixed boundary condition (2), which has resulted in the non-locality in the boundary condition with respect to the separation variable. From the condition (3) we have

$$u_\nu(\omega)\,|_{\theta=\theta_1} = 0\,. \tag{14}$$

We look for the solution of the problem (12)–(14) in terms of the spherical functions. The desired solution of Eq. (12) is taken in the form

$$u_\nu(\omega) = \mathrm{e}^{-\mathrm{i}n\varphi} \left(A_n(\nu) \frac{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta)}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)} + B_n(\nu) \frac{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta)}{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)} \right), \tag{15}$$

Here $P^{-|n|}_{\nu-\frac{1}{2}}(\cdot)$ is the associated Legendre function; see e.g. [16], formula 8.704. Any linear combination of solutions (15) with a sum over n can be also considered as a solution of the homogeneous problem.

The coefficients $A_n(\cdot)$ and $B_n(\cdot)$ are still unknown and should be found from the boundary conditions. Making use of the condition (14), we arrive at

$$A_n(\nu) + B_n(\nu) = 0. \tag{16}$$

The condition (13) leads to

$$A_n(\nu+1) \frac{\mathrm{d}_\theta P^{-|n|}_{\nu+\frac{1}{2}}(\cos\theta)|_{\theta=\frac{\pi}{2}}}{P^{-|n|}_{\nu+\frac{1}{2}}(\cos\theta_1)} + B_n(\nu+1) \frac{\mathrm{d}_\theta P^{-|n|}_{\nu+\frac{1}{2}}(-\cos\theta)|_{\theta=\frac{\pi}{2}}}{P^{-|n|}_{\nu+\frac{1}{2}}(-\cos\theta_1)}$$

$$-K \left(A_n(\nu) \frac{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta)|_{\theta=\frac{\pi}{2}}}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)} + B_n(\nu) \frac{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta)|_{\theta=\frac{\pi}{2}}}{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)} \right) = 0\,, \tag{17}$$

where the notation

$$\mathrm{d}_\theta P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) := \frac{\mathrm{d}}{\mathrm{d}\theta} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) = -\sin\theta \, \frac{\mathrm{d}}{\mathrm{d}x} P^{-|n|}_{\nu-\frac{1}{2}}(x)\bigg|_{x=\cos\theta}$$

is used.

By means of Eq. (16) we eliminate $B_n(\nu) = -A_n(\nu)$ from Eq. (17) thus obtain

$$A_n(\nu+1)\,\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu+\frac{1}{2}}(x)\Big|_{x=0}\left(\frac{(-1)}{P^{-|n|}_{\nu+\frac{1}{2}}(\cos\theta_1)}+\frac{(-1)}{P^{-|n|}_{\nu+\frac{1}{2}}(-\cos\theta_1)}\right)$$

$$-\,KA_n(\nu)\,P^{-|n|}_{\nu-\frac{1}{2}}(0)\left(\frac{1}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)}-\frac{1}{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}\right)=0\,. \tag{18}$$

The solution $A_n(\cdot)$ of the functional difference equation is sought in a class of meromorphic functions, regular in the strip $-C < \Re(\nu) < C$ with some positive C. The behavior of $A_n(\nu)$ as $\nu \to \pm i\infty$ in this strip should be prescribed such that the integral in (9) converges.

4. Solution of the functional equation (18) in an appropriate class of functions

In Eq. (18) it is useful to introduce the new unknown function $a_n(\cdot)$ by the formula

$$a_n(\nu)=A_n(\nu)\,\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)\Big|_{x=0}\left(\frac{(-1)}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)}+\frac{(-1)}{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}\right).$$

The Eq. (18) is then written in the form

$$a_n(\nu+1)-K\,a_n(\nu)\left(\frac{P^{-|n|}_{\nu-\frac{1}{2}}(0)}{\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)|_{x=0}}\right)\left(\frac{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)-P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)+P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}\right)=0\,. \tag{19}$$

Meromorphic solution of Eq. (19) can be found in the form of the product

$$a_n(\nu)=\alpha_n(\nu)\,\beta_n(\nu)\,\gamma_n(\nu), \tag{20}$$

where the meromorphic functions $\alpha_n(\cdot),\ \beta_n(\cdot),\ \gamma_n(\cdot)$ in (20) satisfy the equations

$$\alpha_n(\nu+1)=K\,\alpha_n(\nu)\,, \tag{21}$$

$$\beta_n(\nu+1)=-\,\beta_n(\nu)\left(\frac{P^{-|n|}_{\nu-\frac{1}{2}}(0)}{\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)|_{x=0}}\right), \tag{22}$$

$$\gamma_n(\nu+1)=\gamma_n(\nu)\left(\frac{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)-P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)}{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)+P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}\right) \tag{23}$$

correspondingly. It is obvious that Eq. (21) has an entire solution

$$\alpha_n(\nu)=\exp(\nu\log K)$$

such that $|\alpha_n(\nu)| \le$ Const as $\nu \in (-i\infty, i\infty)$, $|\nu| \to \infty$.

The Eq. (22) is represented in the following way

$$\frac{\beta_n(\nu+1)}{\beta_n(\nu)}=-\frac{P^{-|n|}_{\nu-\frac{1}{2}}(0)}{\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)|_{x=0}}=\frac{1}{2}\,\frac{\Gamma\left(\frac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\frac{-\nu+|n|+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{\nu+|n|+\frac{3}{2}}{2}\right)\Gamma\left(\frac{-\nu+|n|+\frac{3}{2}}{2}\right)}, \tag{24}$$

where we have exploited the expressions [16] (formulas 8.756) written in the form

$$P^{-|n|}_{\nu-\frac{1}{2}}(0)=\frac{2^{-|n|}\sqrt{\pi}}{\Gamma\left(\frac{\nu+|n|-\frac{1}{2}}{2}+1\right)\Gamma\left(\frac{-\nu+|n|+\frac{3}{2}}{2}\right)},$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)\right|_{x=0} = \frac{2^{1-|n|}(-\sqrt{\pi})}{\Gamma\left(\frac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\frac{-\nu+|n|+\frac{1}{2}}{2}\right)},$$

and $\Gamma(\cdot)$ is Euler's gamma-function. By means of the functional equation for the gamma-function $\Gamma(z+1) = z\Gamma(z)$ one can represent an obvious solution regular in the strip $|\Re(\nu)| < \frac{1}{2}$ as

$$\beta_n(\nu) = \frac{\left[\Gamma\left(\frac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\frac{-\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\nu-|n|+\frac{1}{2}\right)\right]^{-1}}{\cos(\pi\nu)}, \tag{25}$$

where $\frac{1}{\cos(\pi\nu)}$ is a meromorphic solution of the equation $\beta(\nu+1) = -\beta(\nu)$. We used it in (25) in order to compensate the exponential growth of the inverse to the product of the gamma-functions so that

$$|\beta_n(\nu)| = O\left(\sqrt{|\nu|}\right) \quad \text{as } \nu \to \pm\mathrm{i}\infty,$$

$|\Re(\nu)| < \frac{1}{2}$. The singularities of $\beta_n(\cdot)$ are real poles which are due to the zeros of $\cos(\pi\nu)$.

Now we consider the solution of Eq. (23) and introduce an auxiliary function $\sigma_n(\cdot)$ by the equality

$$\gamma_n(\nu) = \sigma_n\left(\nu - \frac{1}{2}\right). \tag{26}$$

4.1. Study of the auxiliary equation for $\sigma_n(\cdot)$

The auxiliary equation reads

$$\sigma_n\left(\nu+\frac{1}{2}\right) = \frac{g_n^-(\nu)}{g_n^+(\nu)}\sigma_n\left(\nu-\frac{1}{2}\right), \tag{27}$$

$$g_n^\pm(\nu) = P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1) \pm P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1).$$

Solution of (27) is sought in a class of meromorphic functions which are regular in some strip $|\Re(\nu)| < \nu_*^{|n|}(\theta_1)$ with some positive $\nu_*^{|n|}(\theta_1)$ and is bounded as $|\nu| \to \infty$ in this strip. It is worth noting that the desired solution will be found explicitly, i.e. in quadratures. It is worth remarking that similar equation but with the coefficient depending on trigonometric functions was considered in the work [5]. In our case we deal with the 3D problem, however, in view of the rotational symmetry, one might expect that it resembles 2D problem in an angle with the corresponding boundary conditions on its sides like in [5]. Nevertheless, the coefficient in Eq. (27) depends on a special non-elementary function, i.e. on the associated Legendre function, which is due to the rotationally symmetric geometry of the 3D domain W.

It is sufficient to have such a solution in the strip $|\Re(\nu)| < \frac{1}{2}$ then its meromorphic continuation on the complex plane is performed by means of the functional equation (27). To this end, we study some properties of the entire even functions $g_n^\pm(\cdot)$, in particular, their zeros. Indeed, in order to solve Eq. (27) we intend to take logarithm of both sides and obtain a simple difference equation for $\log\sigma_n(\cdot)$ with the inhomogeneity term $\log\frac{g_n^-(\nu)}{g_n^+(\nu)}$. The latter function requires a definition of its holomorphic branch, which can be done in a traditional way provided that zeroes and poles of the ratio $\frac{g_n^-(\nu)}{g_n^+(\nu)}$ are known. Moreover, asymptotics of the velocity potential as $Kr \to 0$ (and as $Kr \to \infty$) are specified by the position of these poles.

To this end, introduce $Z_n^\pm(x) = P^{-|n|}_{\nu-\frac{1}{2}}(-x) \pm P^{-|n|}_{\nu-\frac{1}{2}}(x)$. Consider $Z_n^+(x)$ which solves the boundary-value problem for the associated Legendre equation ($|n| = 1, 2, \ldots, x = \cos\theta$)

$$-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}Z_n^+(x) + \frac{|n|^2}{1-x^2}Z_n^+(x) = \lambda_\nu^+ Z_n^+(x), \quad x \in (0, x_1),\; x_1 = \cos\theta_1,$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}Z_n^+(x)\right|_{x=0} = 0, \qquad Z_n^+(x)\big|_{x=x_1} = 0.$$

Here $\lambda_\nu^+(|n|) = \nu_+^2(|n|) - \frac{1}{4}$ is an eigenvalue depending on $|n|$. Studying the quadratic form of the self-adjoint operator corresponding to this boundary-value problem with the discrete positive spectrum, we find $\left(\mathcal{L}^+ = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right)$

$$\int_0^{x_1}\mathcal{L}^+Z_n^+(x)\overline{Z_n^+(x)}\,\mathrm{d}x = \int_0^{x_1}(1-x^2)\left|\frac{\mathrm{d}Z_n^+(x)}{\mathrm{d}x}\right|^2\mathrm{d}x =$$

$$\int_0^{x_1}\left[(\nu_+^2 - \frac{1}{4})\left|Z_n^+(x)\right|^2 - \frac{|n|^2}{1-x^2}\left|Z_n^+(x)\right|^2\right]\mathrm{d}x.$$

and conclude that

$$(\nu_m^+(|n|))^2 \;\geq\; \frac{1}{4} \;+\; |n|^2 \;+\; |n|^2 \int_0^{x_1} \frac{x^2\,\left|Z_n^+(x)\right|^2}{1-x^2}\,\mathrm{d}x \left[\int_0^{x_1} \left|Z_n^+(x)\right|^2 \mathrm{d}x\right]^{-1}.$$

Now $\nu_m^+(|n|)$ is a root of the equation

$$\left.(P^{-|n|}_{\nu-\frac{1}{2}}(-x)+P^{-|n|}_{\nu-\frac{1}{2}}(x))\right|_{x=\cos\theta_1} \;=\; 0,$$

therefore, because $\lambda_\nu^+(|n|)>0$, it is on the real axis and subjected to the condition

$$|\,\nu_m^+(|n|)| \;>\; \sqrt{\frac{1}{4}\;+\;|n|^2}.$$

In the same manner, one conclude that the zeros $\nu_m^-(|n|)$ of the equation

$$\left.(P^{-|n|}_{\nu-\frac{1}{2}}(-x)-P^{-|n|}_{\nu-\frac{1}{2}}(x))\right|_{x=\cos\theta_1} \;=\; 0$$

are real and such that

$$|\,\nu_m^-(|n|)| \;\geq\; \sqrt{\frac{1}{4}\;+\;|n|^2}.$$

Remark that $P_{\nu-\frac{1}{2}}(\cos\theta)=1-(\nu^2-1/4)\sin^2(\theta/2)+\cdots$ (see [16], 8.841) and we observe that $\nu=\pm 1/2$ are the roots of the equation $\left.(P_{\nu-\frac{1}{2}}(-x)-P_{\nu-\frac{1}{2}}(x))\right|_{x=\cos\theta_1} \;=\; 0.$

In order to solve Eq. (27) we consider

$$t_n(\nu+\frac{1}{2})-t_n(\nu-\frac{1}{2}) \;=\; \log\left\{\frac{g_n^-(\nu)}{g_n^+(\nu)}\right\}, \tag{28}$$

where

$$t_n(\nu) \;=\; \log\sigma_n(\nu).$$

For any ν from the strip $|\Re(\nu)| \;<\; C_n$ we fix the branch of the function

$$\log\left\{\frac{g_n^-(\nu)}{g_n^+(\nu)}\right\}$$

by the condition

$$\log\left\{\frac{g_n^-(\nu)}{g_n^+(\nu)}\right\} \;=\; O(\exp[-\mathrm{i}\nu(\pi\;-\;2\theta_1)]), \qquad \nu\to-\mathrm{i}\infty$$

and extend the branch-cuts from the zeros of $g_n^\pm$ to ∞ and $-\infty$ along the real axis so that the function $\log\left\{\frac{g_n^-(\cdot)}{g_n^+(\cdot)}\right\}$ is regular in $|\Re(\nu)| \;<\; C_n$ with $C_n=\min_m\{|\nu_m^+(|n|)|,\,|\nu_m^-(|n|)|\}$. The parameter $|n|$ is taken to be arbitrary.

In order to find the solution of (28) and of (27) we consider

$$L_n(\xi) \;=\; \int_{-\mathrm{i}\infty}^{\xi} \mathrm{d}\tau\;\log\left\{\frac{g_n^-(\tau)}{g_n^+(\tau)}\right\},$$

where the integration is performed along the imaginary axis. $L_n(\cdot)$ is holomorphic in $|\Re(\xi)| \;<\; C_n$ and the integral exponentially converges. In particular, $l_n := L_n(\mathrm{i}\infty)$ is finite. We have

$$L_n(\xi) \;=\; O(\exp[-\mathrm{i}\xi(\pi\;-\;2\theta_1)]), \quad \xi\to-\mathrm{i}\infty$$

$$L_n(\xi) \;=\; l_n \;+\; O(\exp[\mathrm{i}\xi(\pi\;-\;2\theta_1)]), \quad \xi\to\mathrm{i}\infty.$$

We make use of the Appendix B in [3], where solution of a class of functional equations (28) is discussed. We arrive at a particular solution to (27):

$$\sigma_n(\nu) \;=\; \exp(t_n(\nu)) \;=\; \exp\left\{\frac{\pi}{2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{L_n(\xi)\,\mathrm{d}\xi}{\cos^2(\pi[\xi\;-\;\nu])}\right\}, \quad |\Re(\nu)| \;<\; C_n+\frac{1}{2}. \tag{29}$$

The solution (29) is meromorphically continued to the whole complex plane by means of (27). One also has

$$\sigma_n(\nu) \;=\; \sigma_n^\pm(1+o(1)) \quad \nu\to\pm\mathrm{i}\infty,$$

$\sigma_n^\pm$ are some known constants.

Remark. By use of the functional equation (27) one can show that $\sigma_0(\nu-1/2)$ has a pole at $\nu=-1/2$ so that $\left.\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\right|_{\theta=\frac{\pi}{2}} = Ku_\nu(\omega)|_{\theta=\frac{\pi}{2}}$ has a singularity in the strip $-1\le\Re(\nu)\le 0$ for $n=0$, which is forbidden by the construction. As a result, in what follows we assume that $|n|\ne 0$ and $|n|=1,2,\ldots$.

5. A closed form expression for the eigenmodes and their asymptotics

The representation (25) is reduced to an equivalent form

$$\beta_n(\nu) = \frac{(-1)2^{|n|-1}}{\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac12}(x)\right|_{x=0}\frac{\left\{\Gamma\left(\nu-|n|+\frac12\right)\right\}^{-1}}{\cos(\pi\nu)}.$$

Exploiting the identity $\Gamma(\frac12-z)\Gamma(\frac12+z)=\frac{\pi}{\cos(\pi z)}$, from (20) and (25) we determine a countable number of solutions satisfying (1), (2) and (3), (4)

$$U_n(r,\omega)=\frac{\mathrm{e}^{-\mathrm{i}n\varphi}}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}\nu\ [Kr]^{\nu-\frac12}\left(\frac{P^{-|n|}_{\nu-\frac12}(\cos\theta)}{P^{-|n|}_{\nu-\frac12}(\cos\theta_1)}-\frac{P^{-|n|}_{\nu-\frac12}(-\cos\theta)}{P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}\right)\times$$

$$\frac{P^{-|n|}_{\nu-\frac12}(\cos\theta_1)\,\Gamma\left(-\nu+|n|+\frac12\right)\sigma_n\left(\nu-\frac12\right)}{\left[1+P^{-|n|}_{\nu-\frac12}(\cos\theta_1)\,[P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)]^{-1}\right]},\quad |n|=1,2,\ldots, \tag{30}$$

where a constant factor in the integrand has been omitted. The integral is exponentially and absolutely convergent on the imaginary axis as $\theta_1\le\theta\le\pi/2-\epsilon$ for any small $\epsilon>0$. However, convergence is only conventional as $\theta=\pi/2$.

From analysis of the positive poles of the integrand in (30) one has

$$|U_n(r,\omega)|\ \le\ \text{Const}\,[Kr]^{\zeta_n}\quad \text{as}\quad Kr\to 0,$$

where Const does not depend on ω and $\zeta_n\geqslant 1$, $|n|=1,2,\ldots$. Making use of the asymptotic estimates of the Mellin integrals [15] and deforming the contour of integration in (30) appropriately, one can derive the asymptotic expansion for $U_n(r,\omega)$ as $Kr\to 0$. Indeed, taking into account the behavior of the integrand, denoted $\mathcal{W}(\nu,r,\theta)$, in (30) as $\theta=\pi/2$ and $\nu\to\mathrm{i}\infty$, $\mathrm{Re}\nu\geqslant 0$, namely,

$$\mathcal{W}(\nu,r,\pi/2)=\mathrm{C_n}\frac{\exp(\nu\log(Kr))}{r^{1/2}|\nu|^{1/2+\mathrm{Re}\nu}}\left(1+O\left(\frac{1}{\nu}\right)\right),$$

we conclude that one can deform the integration contour in (30) from the imaginary axis $\mathrm{i}R$ to $\mathrm{i}R+d$ for some positive d, see [15]. In the process of such deformation some positive poles of the integrand $\mathcal{W}(\nu,r,\pi/2)$ or, more exactly, of

$$\frac{\Gamma\left(-\nu+|n|+\frac12\right)\sigma_n\left(\nu-\frac12\right)}{P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)+P^{-|n|}_{\nu-\frac12}(\cos\theta_1)}$$

can be crossed. These poles are the positive roots of $[\Gamma\left(-\nu+|n|+\frac12\right)]^{-1}$ and of $P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)+P^{-|n|}_{\nu-\frac12}(\cos\theta_1)$.[2] Let ζ_n be the minimal positive root of these functions (n is fixed parameter) then in view of the theorem on residues one has

$$U_n(r,\varphi,\pi/2)\ =\ [\text{Const}_n\,[Kr]^{\zeta_n}\ +O([Kr]^{d_n})]\mathrm{e}^{-\mathrm{i}n\varphi}\quad\text{as}\quad Kr\to 0,$$

where $\zeta_n=\kappa_n-1/2>1/2$ and d_n is equal to the value of the next positive root. The latter asymptotics at the origin is compatible with the Dirichlet boundary condition on the bottom, which means that the potential vanishes at the vertex. Similar asymptotics are also valid as $\theta_1<\theta<\pi/2$, i.e. in the fluid domain.

5.1. *Asymptotic evaluation of the integral (30) as $Kr\to\infty$*

It is convenient to introduce the notation

$$V_n(r,\vartheta)=\ 2\pi\mathrm{i}\sqrt{Kr}\,\mathrm{e}^{\mathrm{i}n\varphi}\,U_n(r,\vartheta,\varphi)=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}I_n(\nu)\mathrm{d}\nu$$

with

$$I_n(\nu)=\mathrm{e}^{\nu\log(Kr)}\left(\frac{P^{-|n|}_{\nu-\frac12}(\cos\theta)}{P^{-|n|}_{\nu-\frac12}(\cos\theta_1)}-\frac{P^{-|n|}_{\nu-\frac12}(-\cos\theta)}{P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}\right)\times$$

[2] Remark that in order to get continuation of $\sigma_n\left(\nu-\frac12\right)$ for any ν with $\Re(\nu)>1/2$ one can use (27).

$$\frac{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\,\Gamma\left(-\nu+|n|+\frac{1}{2}\right)\sigma_n\left(\nu-\frac{1}{2}\right)}{\left[P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)+P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)\right]}.$$

We then introduce the new variable of integration $\mu = \nu(Kr)^{-1}$ and obtain

$$V_n(r,\vartheta) = Kr\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} I_n(Kr\mu)\mathrm{d}\mu =$$

$$Kr\left(\int_{-\mathrm{i}\infty}^{-\mathrm{i}\mu_0} I_n(Kr\mu)\mathrm{d}\mu + \int_{-\mathrm{i}\mu_0}^{\mathrm{i}\mu_0} I_n(Kr\mu)\mathrm{d}\mu + \int_{\mathrm{i}\mu_0}^{\mathrm{i}\infty} I_n(Kr\mu)\mathrm{d}\mu\right),$$

where we split the integration contour onto three parts and chose the constant $\mu_0 > 0$ such that on the semi-infinite parts $[\mathrm{i}\mu_0, \mathrm{i}\infty)$ and $(-\mathrm{i}\infty, -\mathrm{i}\mu_0]$ of the contour one might use the asymptotics of $I_n(Kr\mu)$ as $Kr \to \infty$. We make use of the asymptotics of the associated Legendre functions, as $\nu \to \mathrm{i}\infty$,

$$P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi\sin\theta}}\,\frac{\Gamma(\nu-|n|+1/2)}{\Gamma(\nu+1)}\cos[\nu\theta-\pi|n|/2-\pi/4](1+O(1/\nu)),$$

$\epsilon < \theta < \pi-\epsilon$ for some small $\epsilon > 0$, the Stirling formula for the gamma-function and $\sigma_n(\nu) = \sigma_n^{\pm}(1+O(1/\nu))$ as $\nu \to \pm\mathrm{i}\infty$. Thus we have

$$I_n(Kr\mu) = \frac{\mathrm{e}^{\pm\mathrm{i}(\pi/4-|n|\pi/2)}}{\sqrt{Kr\mu\sin\vartheta}}\,\sigma_n^{\pm}\,\mathrm{e}^{Kr[\mp\mathrm{i}(\vartheta-\pi)\mu+\mu-\mu\log\mu]}(1+O(1/[Kr\mu])),$$

with the upper(lower) signs in the latter expressions taken for $\mu \in [\pm\mathrm{i}\mu_0, \pm\mathrm{i}\infty)$ correspondingly.

In order to calculate the integrals over the semi-infinite parts asymptotically we find the saddle points,[3]

$$G^{\pm}(\mu) = \mp\mathrm{i}(\vartheta-\pi)\mu+\mu-\mu\log\mu,$$

$$\frac{\mathrm{d}G^{\pm}(\mu)}{\mathrm{d}\mu} = \mp\mathrm{i}(\vartheta-\pi)-\log\mu = 0,$$

$$\frac{\mathrm{d}^2G^{\pm}(\mu)}{\mathrm{d}\mu^2} = -\frac{1}{\mu}.$$

As a result, the saddle points are

$$\mu_s^{\pm} = \mathrm{e}^{\pm\mathrm{i}(\pi-\vartheta)}$$

and

$$G^{\pm}(\mu)|_{\mu_s^{\pm}} = \mathrm{e}^{\pm\mathrm{i}(\pi-\vartheta)}, \quad \left.\frac{\mathrm{d}^2G^{\pm}(\mu)}{\mathrm{d}\mu^2}\right|_{\mu_s^{\pm}} = \mathrm{e}^{\pm\mathrm{i}\vartheta}.$$

We can deform the semi-infinite parts of the integration contour in such a manner that they become the steepest descent paths having the equations $\Im\{G^{\pm}(\mu)\} = \Im\{G^{\pm}(\mu_s^{\pm})\}$ and shown in Fig. 2. The whole contour $(-\mathrm{i}\infty, \mathrm{i}\infty)$ is then deformed accordingly into that coinciding with that L drawn by the solid line. In the process of such deformation several negative poles $\nu = -\varkappa_m, (m = 1, 2, \ldots M, \ \ \varkappa_m > 1)$ of the integrand $I_n(\nu)$ are crossed. Remark that, if Kr is large, $M = M(Kr)$ is also large. These negative poles are obviously the singularities of the meromorphic factor

$$\frac{\sigma_n\left(\nu-\frac{1}{2}\right)}{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)+P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)}$$

in the integrand.

Basic contributions of the saddle points are given by the integration over the SD paths in $O([Kr]^{-1/2})$—vicinities of the saddle points. Then, making use of the saddle point technique, we find

$$U_n(r,\vartheta,\varphi) = \frac{\mathrm{e}^{-\mathrm{i}n\varphi}}{\mathrm{i}\sqrt{2\pi\,Kr\sin\vartheta}}\left\{\mathrm{e}^{\mathrm{i}(\pi/4-|n|\pi/2)}\sigma_n^{+}\,\mathrm{e}^{-Kr\cos\vartheta+\mathrm{i}Kr\sin\vartheta}+\right. \tag{31}$$

$$\left.\mathrm{e}^{-\mathrm{i}(\pi/4-|n|\pi/2)}\sigma_n^{-}\,\mathrm{e}^{-Kr\cos\vartheta-\mathrm{i}Kr\sin\vartheta}\right\}(1+O(1/Kr)) + \mathrm{e}^{-\mathrm{i}n\varphi}\sum_{1\le m\le M}\frac{P_m(\vartheta)}{(Kr)^{\varkappa_m+1/2}}+$$

[3] The idea of application of the saddle point technique to the evaluation the Mellin integral at hand is due to remarks of Dr. Eng. Ning Yan Zhu from Stuttgart University.

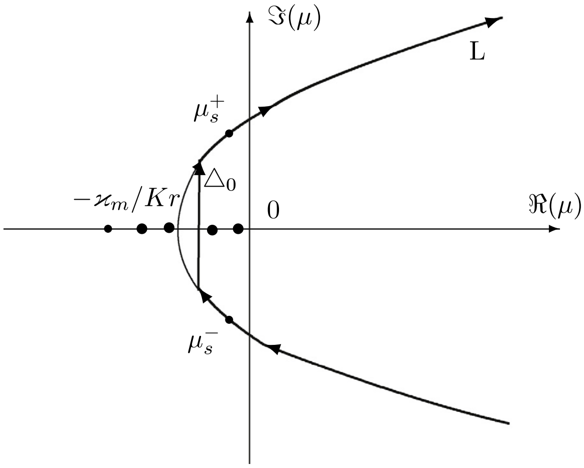

Fig. 2. The steepest descent contour.

$$\frac{\mathrm{e}^{-\mathrm{i}n\varphi}\sqrt{Kr}}{2\pi\mathrm{i}}\int_{\triangle_0} I_n(Kr\mu)\mathrm{d}\mu,$$

where in the last term integration is performed along the vertical segment $\triangle_0$ (Fig. 2) of the contour L. The integral is estimated as $O((Kr)^{-\varkappa_{M+1}-1/2})$. The latter formula is valid for all $\vartheta \in (\pi/2-\delta, \pi/2]$, for some $\delta > 0$. Explicit formulae can be given for $P_m(\vartheta)$ by means of the corresponding residues of the integrand. However, they are omitted here.

It is worth commenting on the asymptotics (31). Provided $\vartheta = \frac{\pi}{2}$, i.e. on the free surface, the first two terms play the leading role. They represent the sum of the outgoing

$$U_n^{\mathrm{out}}(Kr,\varphi) = \frac{\mathrm{e}^{-\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}Kr}}{\mathrm{i}\sqrt{2\pi\,Kr}}\mathrm{e}^{\mathrm{i}(\pi/4-|n|\pi/2)}\sigma_n^+(1+O(1/Kr))$$

and incoming

$$U_n^{\mathrm{in}}(Kr,\varphi) = \frac{\mathrm{e}^{-\mathrm{i}n\varphi}\,\mathrm{e}^{-\mathrm{i}Kr}}{\mathrm{i}\sqrt{2\pi\,Kr}}\mathrm{e}^{-\mathrm{i}(\pi/4-|n|\pi/2)}\sigma_n^-(1+O(1/Kr))$$

surface gravity waves. The outgoing surface wave satisfies the Sommerfeld radiation condition

$$\left(\frac{\partial}{\partial r}-\mathrm{i}K\right)U_n^{\mathrm{out}}(Kr,\varphi) = O([Kr]^{-3/2})$$

on the free surface $\theta = \pi/2$ uniformly w.r.t. φ, whereas the incoming wave $U_n^{\mathrm{in}}(Kr,\varphi)$ does not fulfill this condition.

The presence of U_n^{out} and U_n^{in} in the far-field asymptotics on the free surface ought to be additionally commented by means of the following analogy. Consider Green's function G^{out} of the problem for the water halfspace in 3D when the source is located at the origin O, Section 1.1 in [1], and the solution satisfies the radiation condition. The far field asymptotics, when the observation point is also near the free surface, is (see (1.15) in [1])

$$G^{out} \sim 2\pi\mathrm{i}K\,\mathrm{e}^{-Kr\cos\vartheta}\,H_0^{(1)}(Kr\sin\theta) = C_+\,\frac{\mathrm{e}^{\mathrm{i}Kr\sin\theta-Kr\cos\vartheta}}{\sqrt{Kr}}(1+O(1/Kr)).$$

In the same manner, at large distances the 'incoming' solution G^{in} has the asymptotics

$$G^{in} \sim 2\pi\mathrm{i}K\,\mathrm{e}^{-Kr\cos\vartheta}\,H_0^{(2)}(Kr\sin\theta) = C_-\,\frac{\mathrm{e}^{-\mathrm{i}Kr\sin\theta-Kr\cos\vartheta}}{\sqrt{Kr}}(1+O(1/Kr))$$

if the observation point is near the free surface F. As a result of this simple observation, in the leading approximation we can interpret the wave field (31) near the free surface F as a sum of the incoming and outgoing surface waves from an imaginary point source at the point $O \in F$. The vertex of the conical surface B plays the role of such a virtual source in our case. In this sense, near the free surface F the wave field behavior is universal at large distances from a small scatterer located on the surface of a fluid. However, the excitation coefficients of U_n^{out} and U_n^{in} are determined from the analysis of the specific geometry of the bottom B near the conical vertex O.

It is obvious that the surface integral in the functional of energy $E(U_n)$ for the surface waves in (6) diverges. On the other hand, as $\vartheta < \frac{\pi}{2}$ the first two terms in (31) are exponentially small for $Kr \to \infty$ and the leading contribution is given by the power series in (31). For these terms in the asymptotics (31) the integrals for the energy (6) converge. But the energy is infinite for the total set of summands in (31) as a result of divergence of the surface integral. Such divergence is due to the slow decay of the incoming and outgoing surface waves, which might be foreseen.

As a result, in view of the asymptotic estimates (31) the eigenfunctions $U_n, \quad |n| = 1, 2, \ldots$ correspond to the *continuous* spectrum, assuming that we accept the definition for the eigenmodes of the continuous spectrum given in Section 2.[4] It is obvious that the multiplicity of the spectrum is infinite. Some additional aspects of the spectral problems in infinite domains are discussed in [17,18]. On the other hand, some general physical conceptions [19], based on the radiation stress of the water waves, could be useful in further physical interpretations of the results in our study, see also [20].

6. Conclusion

In this work we have constructed eigenfunctions of the continuous spectrum for the problem of the eigenoscillations of a fluid in a pool of special canonical form. To this end some new functional equations have been studied and explicitly solved. The far field asymptotics has been computed from the corresponding Mellin integral representation of the eigenfunctions and some physical analysis of the solution has been also given.

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