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ORDINARY DIFFERENTIAL EQUATIONS

Bifurcation of the Equilibrium of an Oscillator with a Velocity-Dependent Restoring Force under Periodic Perturbations

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Abstract— We study the bifurcation of an oscillator whose restoring force depends on the velocity of motion under periodic perturbations. Separation of variables is used to derive a bifurcation equation. To each positive root of this equation, there corresponds an invariant two-dimensional torus (a closed trajectory in the case of a time-independent perturbation) shrinking to the equilibrium position as the small parameter tends to zero. The proofs use methods of the Krylov–Bogolyubov theory for the case of periodic perturbations or the implicit function theorem for the case of time-independent perturbations.

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1. STATEMENT OF THE PROBLEM

Consider the differential equation

$$\ddot{x} + x^{p/q} + ax^m\dot{x} = X(x, \dot{x}, t, \varepsilon),$$

which will be viewed as a t-periodic perturbation of the equation

$$\ddot{x} + x^{p/q} + ax^m\dot{x} = 0.$$

(The conditions imposed on the numbers p, q, a, m, and ε and the function X are given below.)

It is expedient to assign the dimension q to the variable x; then the dimension of $x^{p/q}$ is p. Let us require that the dimensions of all three terms in the unperturbed equation be the same, i.e., p. If we assign a dimension k to the variable $y = \dot{x}$, then, since the dimension of x is q, it follows that the dimension of the derivative of a function is obtained by adding $k - q$ to the dimension of the function itself. Therefore, the dimension of the variable $\dot{y} = \ddot{x}$ is $2k - q$, and the above-indicated condition takes the form of the relations $p = 2k - q = mq + k$. Thus,

$$k = (p + q)/2, \qquad m = (p - q)/(2q).$$

Further, the dimension of the function $X(x, \dot{x}, t, \varepsilon)$ must not be lower than $p + 1$.

Thus, we consider the differential equation

$$\ddot{x} + x^{p/q} + ax^{(p-q)/(2q)}\dot{x} = X(x, \dot{x}, t, \varepsilon) \tag{1}$$

under the following assumptions 1–4.

1. The numbers p and q are odd and coprime (in particular, p/q is an irreducible fraction), $p > q > 1$, and $\varepsilon > 0$ is a small parameter.

Let us divide the set of numbers p into residue classes modulo $4q$. In other words, we set $p = \widetilde{p} + 4q\ell$, where $\ell = 0, 1, 2, \ldots$ and $\widetilde{p} \in \{q + 2, \ldots, 5q - 2\}$ is odd.

The present paper deals with the case in which

$$4q + 1 \le \widetilde{p} \le 5q - 2. \tag{2}$$

The other cases can be studied in a similar way (see Section 4).

2. The function $X(x, y, t, \varepsilon)$ is sufficiently smooth in (x, y, ε) in a neighborhood of the point $(x, y, \varepsilon) = (0, 0, 0)$, is continuous and periodic in t (in particular, it may be independent of t), and satisfies $X(0, 0, t, \varepsilon) = 0$. We denote the period by ω.

3. The expansion of $X(x, y, t, \varepsilon)$ in powers of x, y, and ε does not contain terms of order lower than $p + 1$, where x has order q, y has order $(p + q)/2$, and ε has order $2(1 + \ell)q$.

4. The inequality $0 < a^2 < 2(p + q)/q$ is satisfied.

Consider the unperturbed equation (1), i.e., the equation

$$\ddot{x} + x^{p/q} + ax^{(p-q)/(2q)}\dot{x} = 0. \tag{3}$$

The stability of the zero solution for $q = 1$ under time-independent perturbations without the small parameter was studied by Lyapunov [1].

Equation (3) is equivalent to the system

$$\dot{x} = y, \qquad \dot{y} = -x^{p/q} - ayx^{(p-q)/(2q)}, \tag{4}$$

whose trajectories are determined by the differential equation

$$y\,dy + (x^{p/q} + ax^{(p-q)/(2q)}y)\,dx = 0.$$

We integrate this equation using the substitution $y = zx^{(p+q)/(2q)}$ and obtain

$$\frac{p+q}{2}y^2 + aqx^{(p+q)/(2q)}y + qx^{(p+q)/q} = D\exp\left\{\frac{2a}{\sqrt{\Delta}}\arctan\frac{q^{-1}(p+q)y + ax^{(p+q)/(2q)}}{x^{(p+q)/(2q)}\sqrt{\Delta}}\right\}, \tag{5}$$

where D is a positive constant and $\Delta = 2(p + q)/q - a^2$. By condition 4, we have $\Delta > 0$.

Since $\Delta > 0$, it follows that the function on the left-hand side is positive definite. If, in addition, $(p + q)/2$ is even, then the curves defined by Eq. (5) are closed and surround the origin.

Equation (1) is equivalent to the system

$$\dot{x} = y, \qquad \dot{y} = -x^{p/q} - ayx^{(p-q)/(2q)} + X(x, y, t, \varepsilon). \tag{6}$$

There are only two possible cases: (a) the number $(p + q)/2$ is even, or, equivalently, $(p - q)/2$ is odd; (b) the number $(p + q)/2$ is odd, or, equivalently, $(p - q)/2$ is even.

It has been noted that the singular point $(0, 0)$ of system (4) is a center in case (a). In case (b), it is a focus.

Note that if $\Delta \le 0$, then the singular point $(0, 0)$ of system (4) is a node (stable for $a > 0$ and unstable for $a < 0$) for odd $(p + q)/2$ and a saddle point for even $(p + q)/2$.

Thus, Eq. (3) is an oscillator only if $\Delta > 0$ and $(p + q)/2$ is even. It is this case that will be considered below.

The present paper provides conditions under which, in addition to the equilibrium position $x = 0$, Eq. (1) has an invariant two-dimensional torus (a closed trajectory under a time-independent perturbation) for each sufficiently small $\varepsilon > 0$. The case of $q = 1$ was studied in [2], and the case of $a = 0$, in [5].

2. BIFURCATION EQUATION

Let us introduce functions $C(\varphi)$ and $S(\varphi)$ that satisfy system (4); i.e.,

$$C'(\varphi) = S(\varphi), \qquad S'(\varphi) = -C^{p/q}(\varphi) - aSC^{(p-q)/(2q)}(\varphi). \tag{7}$$

For $q = 1$ and $a = 0$, the functions C and S were considered by Lyapunov [1].