Chiral transitions in three-dimensional magnets and higher order $\epsilon$ expansion

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Abstract

The critical behaviour of helimagnets and stacked triangular antiferromagnets is analyzed in $4 - \epsilon$ dimensions within the three-loop approximation. Numerical estimates for marginal values of the order parameter dimensionality $N$ obtained by resummation of corresponding $\epsilon$-expansions rule out the possibility of continuous chiral transitions in magnets with Heisenberg or planar spins.

Keywords: $\epsilon$-expansion; Chiral transitions; Helimagnets; Stacked triangular antiferromagnets

Chiral phase transitions in helical magnets and stacked triangular antiferromagnets with Heisenberg or XY-like spins as well as in some other systems have attracted much attention during the last decade [1-9]. The special interest in these transitions, both for theorists and by experimentalists, is due to the conjecture [1-3] that they belong to a new universality class, which is characterized by critical exponents differing markedly from those of 3D O(n)-symmetric model with relevant values of $n$ (2, 4, and 6). This conjecture originates from the renormalization-group (RG) analysis of the corresponding $(4 - \epsilon)$-dimensional model performed within the lowest (one- and two-loop, according to the quantum field theory language) orders in $\epsilon$. Some results given by the $(1/n)$-expansion and the Monte Carlo simulations were also considered as favoring the abovementioned idea [2,3].

In this Letter an attempt is made to clear up whether the conclusion of the existence of a new universality class for 3D chiral systems with Heisenberg or planar spins survives when higher-order terms in the $\epsilon$-expansion are taken into account. Below we calculate $\epsilon$-expansions for quantities of interest up to the three-loop order. To obtain numerical estimates relevant to real 3D magnets we apply resummation procedures to these series before setting $\epsilon = 1$. Such a machinery proved to give good results for many phase transition models. It is believed to be powerful enough to yield reasonable predictions in our case as well.

The Landau–Wilson Hamiltonian describing the systems under consideration may be written in the form (see, e.g., Ref. [9])

$$H = \frac{1}{2} \int d^3x \left[ m_0^2 \varphi_\alpha \varphi_\alpha^* + \nabla \varphi_\alpha \nabla \varphi_\alpha^* + \frac{1}{2} m_0 \varphi_\alpha \varphi_\alpha \varphi_\beta \varphi_\beta + \frac{1}{2} m_0 \varphi_\alpha \varphi_\alpha \varphi_\beta \varphi_\beta \right],$$

(1)

where $\varphi_\alpha$ is a complex vector order parameter field, $\alpha, \beta = 1, 2, \ldots, N$, the bare mass squared, $m_0^2$, is proportional to the deviation from the mean-field transition point. This model undergoes chiral phase transi-
In the opposite case, the transitions into somewhat trivial (linearly polarized or unfrustrated) ordered states occur.

In the critical region, where fluctuations are strong and the system behavior is governed by the RG equations, the model equation (1) can demonstrate four different regimes of RG flow depending on $N$ \[^{[3,9]}\]. Correspondingly, three critical (marginal) values of $N$ exist separating these regimes from each other. If $N < N_{c1}$ the RG equations possess three nontrivial fixed points (FPs) with the $O(2N)$-symmetric point being stable. When $N$ exceeds $N_{c1}$ the Heisenberg FP loses its stability but the other, anisotropic FP with coordinate $w < 0$ acquires it. This point "annihilates" with another, saddle anisotropic FP when $N$ approaches $N_{c2}$, and there is only one nontrivial FP in the domain $N_{c2} < N < N_{c3}$. It is $O(2N)$-symmetric and unstable. At last, when $N$ increases further and passes the value $N_{c3}$ the creation of two new anisotropic FPs with $w > 0$ occurs. One of them is stable and describes the chiral critical behavior. Hence, to answer the question of the relevance of the chiral FP to the critical thermodynamics of real helical magnets and stacked triangular antiferromagnets, one has to estimate $N_{c3}$ and compare the value obtained with the physical values $N = 2$ and $N = 3$.

Marginal values of $N$ may be found analyzing the RG $\beta$-functions. We calculate these functions for the model equation (1) within the three-loop approximation in $4 - \varepsilon$ dimensions using the minimal subtraction scheme (corresponding expansions are too lengthy and not presented here). The $\varepsilon$-expansion for $N_{c3}$ resulting from the $\beta$-functions obtained is as follows,

$$
N_{c3} = 12 + 4\sqrt{6} - (12 + \frac{14}{3} \sqrt{6}) \varepsilon
+ \left[ \frac{1 + \frac{3}{2} \sqrt{6}}{180} + \frac{91}{300} \sqrt{6} + \frac{13}{5} \right] \varepsilon^2
= 21.80 - 23.43 \varepsilon + 7.088 \varepsilon^2,
$$

where $\zeta(x)$ is the Riemann $\zeta$-function, $\zeta(3) = 1.20206$. The constant and linear terms in Eq. (2) coincide with those presented by Kawamura \[^{[3]}\] while the second-order term is essentially new.

Such expansions are known to be asymptotic, and physical information may be extracted from them provided some resummation method is applied. The Borel transformation combined with a proper procedure of analytical continuation of the Borel transform usually plays the role of this method leading to precise numerical estimates in cases of a sufficiently long original series \[^{[10,11]}\]. To perform the analytical continuation the Padé approximant of $[1/1]$ type may be used, which is known to provide rather good results for various Landau–Wilson models (see, e.g., Refs. \[^{[9,12,13]}\]). The Padé–Borel summation of the expansion (2) gives

$$
N_{c3} = a - \frac{2b^2}{c} + \frac{4b^3}{c^2} \exp\left( -\frac{2b}{c} \right) \text{Ei}\left( \frac{2b}{c} \right),
$$

where $a$, $b$, and $c$ are the coefficients before $\varepsilon^0$, $\varepsilon^1$, and $\varepsilon^2$ in Eq. (2), respectively, $\text{Ei}(x)$ being the exponential integral. Setting $\varepsilon = 1$, we obtain from Eq. (3)

$$
N_{c3} = 3.39.
$$

Making use of the Padé approximant $[1/1]$ itself gives $N_{c3} = 3.81$ while direct summation of the expansion equation (2), being a rather crude procedure, results in $N_{c3} = 5.46$.

All these values, although considerably scattered, are nevertheless greater than 3. Hence, helical magnets and stacked triangular antiferromagnets with Heisenberg and $XY$-like spins are not seen to demonstrate new, chiral critical behavior. Instead, they should approach the helically ordered state or frustrated antiferromagnetic phase only via first-order phase transitions.

On the other hand, the difference between the number equation (4) and $N = 3$ is not so large. Moreover, numerical estimates for $N_{c3}$ were obtained from the theory having no small parameter in the limit $\varepsilon = 1$. How close to the precise value of $N_{c3}$ may they be?

To clear up this point let us compare Eq. (4) and its Padé counterpart with an analogous estimate given by the RG analysis in three dimensions. Three-loop RG expansions for $\beta$-functions resummed by the generalized Padé-Borel technique result in $N_{c3} = 3.91$ \[^{[9]}\]. Since the accuracy provided by this approximation was argued to be rather high \[^{[9]}\] (about 1% for FP coordinates and critical exponents), the resummed $\varepsilon$-expansion, within three-loop order, seems to yield plausible estimates for the quantities of interest.

This conclusion is of prime importance. It is reasonable, therefore, to examine it further, e.g. by calculating the rest of the marginal values of $N$ from the resummed three-loop $\varepsilon$-expansions and by making
a comparison of the values obtained with the corresponding 3D RG estimates. The inequality $N_{c2} > 2$ proven earlier [9] may be used as a criterion in the course of such a study. The $\epsilon$-expansions for $N_{c1}$ and $N_{c2}$ are found to be

$$N_{c1} = 2 - \epsilon + \frac{\epsilon^2}{2^2} \left[ 6 \xi(3) - 1 \right] \epsilon^2$$

$$= 2 - \epsilon + 1.294 \epsilon^2,$$

(5)

$$N_{c2} = 12 - 4\sqrt{6} - \left( 12 - \frac{12}{3} \sqrt{6} \right) \epsilon$$

$$+ \left[ \frac{137}{150} - \frac{91}{100} \sqrt{6} + \left( \frac{131}{5} - \frac{47}{12} \sqrt{6} \right) \xi(3) \right] \epsilon^2$$

$$= 2.202 - 0.569 \epsilon + 0.989 \epsilon^2.$$

(6)

Their Padé-Borel summation gives for $\epsilon = 1$

$$N_{c1} = 1.50, \quad N_{c2} = 1.96.$$  

(7)

These values are close to those obtained in 3D, $N_{c1} = 1.45, N_{c2} = 2.03$ [9], but $N_{c2}$ is obviously underestimated by the $\epsilon$-expansion since the corresponding value is less than 2. The difference $2 - N_{c2}^{(3)} = 0.04$, however, is small and may be considered as a lower bound for the error produced by this approximation. The most likely estimate for this error is believed to be close to $N_{c}^{(3D)} - N_{c}^{(\epsilon)}$, i.e., of the order of 0.1.

For $N_{c1}$ the $\epsilon$-expansion predicts a value which is slightly greater than $N_{c1}^{(3D)}$. At the same time, these values differ from each other by only 3% and lie so far from the nearest physical value $N = 2$ that this difference is quite unimportant.

We see that the resummed three-loop $\epsilon$-expansion gives sufficiently good numerical estimates for $N_{c1}$ and $N_{c2}$ providing, however, a lower accuracy when used for the evaluation of the highest critical dimensionality $N_{c3}$. This is not surprising since the structure of Eq. (2) turns out to be rather unsuitable for yielding reliable quantitative results. Indeed, to obtain precise numerical estimates one has to deal with a series which, at least, possess coefficients decreasing with increasing value. Instead, the second term in Eq. (2) exceeds, for $\epsilon = 1$, the first one. It is clear that such an expansion would not demonstrate a good summability. That is why we believe that the true value of $N_{c3}$ is closer to the 3D estimate 3.91 than to 3.39. On the other hand, the latter estimate differs from the former by no more than 15%. Hence, the current status of the $\epsilon$-expansion is not so bad provided three-loop contributions are properly taken into account. It seems natural that calculations of higher-order terms will result in a further improvement of the numerical estimates.

It is worth noting that in fact the three-loop terms added and the summation have changed the situation drastically. The point is that the two-loop $\epsilon$-expansions for $N_{c1}, N_{c2},$ and $N_{c3}$ directly extrapolated to $\epsilon = 1$ violate the inequalities $N_{c1} < N_{c2} < N_{c3}$, which should hold according to the definition of $N_{c2};$ in this approximation $N_{c3}^{(\epsilon)} < N_{c2}^{(\epsilon)} < N_{c1}^{(\epsilon)}$ when $\epsilon = 1$. This is a warning that with such a short series in hand one cannot safely penetrate into the three-dimensional world. On the other hand, the values given by the Padé and Padé-Borel resummed three-loop $\epsilon$-expansions at $\epsilon = 1$ meet the abovementioned inequalities.

Another point to be specially marked is the following. We have now enough information resulting from higher-order RG analysis in three and $4 - \epsilon$ dimensions to draw a firm conclusion on the critical behavior of the $XY$-like systems. Indeed, since $N_{c2}$ has been proven to be greater than 2 [9] and $N_{c3}$ should exceed $N_{c2},$ magnets with planar spins certainly cannot undergo continuous chiral transitions. Only first-order phase transitions into the chiral states are possible, in principle, in these systems.

We conclude with a comment concerning the $(1/n)$-expansion and the Monte Carlo simulations. Their predictions do not contradict those just obtained. Indeed, as was shown in Ref. [13], even for a simple, $O(n)$-symmetric model the $(1/n)$-expansion begins to yield reasonable numerical estimates only when $n$ exceeds 20. Since for the systems studied $n = 2N = 4, 6,$ this approach is obviously inapplicable in our case. Monte Carlo simulations, in turn, give values for the critical exponents which are close to the tricritical ones, especially for systems with $XY$-like spins [14]. That is why it is believed [5, 9, 14] that not the chiral critical behavior but the tricritical or the tricritical-to-critical crossover have in fact been seen in these computer experiments.

To summarize, we have found three-loop contributions to the $\epsilon$-expansions of the critical order-parameter dimensionality $N_{c1}, N_{c2},$ and $N_{c3}$ for the model describing the chiral critical behavior. The Padé-Borel summation of the series obtained has yielded, at $\epsilon = 1,$ fair numerical estimates for $N_{c1}$ and $N_{c2}$ in 3D. For the lower boundary of the domain where continuous chiral transitions are possible the $\epsilon$-expansion resummed by the Padé-Borel and
the simple Padé methods has given $N_{c3} = 3.39$ and $N_{c3} = 3.81$, respectively. Being sufficiently close to the 3D RG estimate $N_{c3} = 3.91$, these values are larger than the physical values $N = 2$ and $N = 3$. This may be considered as evidence that in magnets with Heisenberg or planar spins the chiral critical behavior with specific values of the critical exponents in fact would not occur and that they cannot belong to a new, chiral class of universality. Since $N_{c3} > N_{c2}$ and $N_{c2}$ should exceed 2, for the XY-like systems this conclusion sounds firm.

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