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MATHEMATICS

Stable and Completely Unstable Periodic Points of Diffeomorphism of a Plane with a Heteroclinic Contour

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Abstract—The diffeomorphism of a plane into itself with three hyperbolic points is studied this paper. It is assumed that the heteroclinic points lie at the intersections of the unstable manifold of the first point and the stable manifold of the second point, of the unstable manifold of the second point and the stable manifold of the third point, of the unstable manifold of the third point and the stable manifold of the first point. The orbits of fixed and heteroclinic points form a heteroclinic contour. The case when stable and unstable manifolds intersect non-transversally at heteroclinic points is investigated. The points of tangency of finite order are firstly distinguished among the points of non-transversal intersection of a stable manifold with an unstable manifold; in this paper, such points are not considered. Diffeomorphism with a heteroclinic contour was studied in the works of L.P. Shilnikov, S.V. Gonchenko, and other authors, and it was assumed that the points of non-transversal intersection of stable and unstable manifolds are points of tangency of finite order. It follows from the works of these authors that a diffeomorphism exists for which there are stable and completely unstable periodic points in the neighborhood of the heteroclinic contour. It is assumed in the current paper that the points of non-transversal intersection of stable and unstable manifolds are not the points of tangency of finite order. It is demonstrated that two countable sets of periodic points may lie in the neighborhood of such a heteroclinic contour. One of these sets consists of stable periodic points whose characteristic exponents are separated from zero, and another set consists of completely unstable periodic points whose characteristic exponents are also separated from zero.

Keywords: diffeomorphism of plane, hyperbolic fixed points, heteroclinic points, heteroclinic contour, non-transversal intersection, stability.

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1. INTRODUCTION

In this work we study diffeomorphisms of a plane into itself of the class C^r $(1 \le r \le \infty)$ with a non-transversal heteroclinic contour. The work is mainly aimed at distinguishing the class of diffeomorphisms which has infinite sets of stable and completely unstable periodic points with characteristic exponents separated from zero in a bounded neighborhood of the contour. A periodic point of diffeomorphism of a plane is stable if its characteristic exponents are negative, and it is hyperbolic (saddle) if these exponents have opposite signs. We call a periodic point completely unstable if its characteristic exponents are positive. An example of diffeomorphism of a plane which has a countable set of stable periodic points with characteristic exponents separated from zero in a bounded part of the plane is provided in [1].

Suppose F is a diffeomorphism of a plane into itself, $F: \mathbb{R}^2 \to \mathbb{R}^2$, and z is a periodic point with period m (at $m = 1$ the point is called fixed). Suppose ρ_1 and ρ_2 are the eigenvalues of the matrix $DF^m(z)$; then, the characteristic exponents of the point z are the values

$$\nu_i = (m)^{-1} \ln|\rho_i|, \quad i = 1, 2. \tag{1}$$

Suppose $z_1, z_2, \ldots, z_n$ are the fixed hyperbolic points of the diffeomorphism F and $W^s(z_1)$, $W^u(z_1)$, ..., $W^s(z_n)$, $W^u(z_n)$ are stable and unstable manifolds of these points; none of these manifolds are trivial. We assume that points $w_1, w_2, \ldots, w_n$ exist such that $w_i \in W^u(z_i) \cap W^s(z_{i+1})$, $i = 1, 2, \ldots, n-1$, $w_n \in W^s(z_1) \cap W^u(z_n)$. The points $w_1, w_2, \ldots, w_n$, $n > 1$, are referred to as heteroclinic points and the union of trajectories of the points $w_1, w_2, \ldots, w_n$ and their limit manifolds are referred to as heteroclinic contour. At $n = 1$ it is said that a hyperbolic fixed point has a point homoclinic to it. A heteroclinic contour is called transversal if in all heteroclinic points stable and unstable manifolds intersect transversally; in the opposite case, the contour is called non-transversal. Similarly, if stable and unstable manifolds intersect in a homoclinic point transversally, then this homoclinic point is referred to as transversal; in the opposite case, it is referred to as non-transversal.

Diffeomorphisms with non-transversal heteroclinic contours have been studied in many works (see, e.g., [2–5]). Tangency of stable and unstable manifolds in a heteroclinic point is determined by the scalar function of two variables expressed through the original diffeomorphism. If a value $l > 1$ exists such that the values of all derivatives of this function with respect to one coordinate are equal to zero until the order $l - 1$ in the heteroclinic point and if the value of the l-order derivative is separated from zero, then the tangency of stable and unstable manifolds is the finite-order tangency. At $l = 2$ the tangency is called quadratic. It is demonstrated in [6] that infinite number of stable periodic points may lie in the neighborhood of a non-transversal homoclinic point with finite order tangency, but at least one characteristic exponent tends to zero as the period increases. The diffeomorphism with non-transversal heteroclinic contour at $n = 2$ was investigated in [3]. It was assumed that the tangency is not higher than quadratic in heteroclinic points. It was shown that infinite number of stable periodic points lies in the neighborhood of the contour, but the characteristic exponents of these points tend to zero as the period increases. It was demonstrated in [4, 5] that, in the neighborhood of a diffeomorphism with non-transversal heteroclinic contour with $n > 1$ whose heteroclinic points have the tangency not higher than quadratic, there lie diffeomorphisms with infinite number of stable and completely unstable periodic points. It was shown in [7] that the neighborhood of a non-transversal homoclinic point, when it is not a point with finite-order tangency, may contain infinite number of stable periodic points with characteristic exponents separated from zero.

In this work we study diffeomorphisms of a plane into itself with non-transversal heteroclinic contour at $n = 3$. We assume the tangencies of stable and unstable manifolds are not finite-order tangencies. We show that countable sets of stable and completely unstable periodic points with characteristic exponents separated from zero may lie in an arbitrary neighborhood of the contour.

2. MAIN THEOREM

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and let the points $O = (0, 0)$, $A = (a_1, a_2)$, and $B = (b_1, b_2)$, where $0 < a_1 < b_1$ and $a_2 < 0 < b_2$, be its hyperbolic fixed points. We assume that bounded neighborhoods V_1, V_2, and V_3 of the points O, A, and B, respectively, exist such that

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu_1 x \\ \lambda_1 y \end{pmatrix}, \tag{2}$$

where $(x, y) \in V_1$,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + \lambda_2(x - a_1) \\ a_2 + \mu_2(y - a_2) \end{pmatrix}, \tag{3}$$

where $(x, y) \in V_2$,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 + \mu_3(x - b_1) \\ b_2 + \lambda_3(y - b_2) \end{pmatrix}, \tag{4}$$

where $(x, y) \in V_3$ and $0 < \lambda_i < 1 < \mu_i$, $i = 1, 2, 3$.

We assume that

$$\lambda_1\mu_2 < 1, \quad \lambda_2\mu_3 < 1, \quad \lambda_3\mu_3 > 1, \tag{5}$$

$$\lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3 < 1. \tag{6}$$

Suppose that $\alpha \in \mathbb{N}$ is such that

$$\lambda_1\mu_1\lambda_2\mu_2(\lambda_3\mu_3)^{\alpha} > 1. \tag{7}$$

We assume that the manifolds $V_1 \cup F(V_1)$, $V_2 \cup F(V_2)$, and $V_3 \cup F(V_3)$ do not mutually intersect.

The local stable and unstable manifolds $W^s_{\text{loc}}(O)$ and $W^u_{\text{loc}}(O)$ lie on the coordinate axes and the manifolds $W^s_{\text{loc}}(A)$, $W^u_{\text{loc}}(A)$, $W^s_{\text{loc}}(B)$, and $W^u_{\text{loc}}(B)$ lie on the straight lines parallel to the axes.

Suppose that the positive p_i, where $i = 1, ..., 6$, are such that $0 < p_1 < b_2$, $0 < p_2 < p_3 < a_1$, $0 < p_4 < p_5 < b_2$, and $0 < p_6 < b_1$. We think that the following relations are valid: $(0, p_1) \in V_1$, $(0, p_1) \notin F(V_1)$, $(p_2, 0) \in F(V_1)$, $(p_2, 0) \notin V_1$, $(p_3, a_2) \in V_2$, $(p_3, a_2) \notin F(V_2)$, $(a_1, p_4) \in F(V_2)$, $(a_1, p_4) \notin V_2$, $(b_1, p_5) \in V_3$, $(b_1, p_5) \notin F(V_3)$, $(p_6, b_2) \in F(V_3)$, and $(p_6, b_2) \notin V_3$. It is clear that $(0, p_1) \in W^s_{\text{loc}}(O)$, $(p_2, 0) \in W^u_{\text{loc}}(O)$, $(p_3, a_2) \in W^s_{\text{loc}}(A)$, $(a_1, p_4) \in W^u_{\text{loc}}(A)$, $(b_1, p_5) \in W^s_{\text{loc}}(B)$, and $(p_6, b_2) \in W^u_{\text{loc}}(B)$.

Suppose that U_1, U_2, and U_3 are such neighborhoods of the points $(p_2, 0)$, (a_1, p_4), and (p_6, b_2) that $U_i \subset F(V_i)$ and $U_i \cap V_i = \emptyset$ and S_1, S_2, S_3 are such neighborhoods of points $(0, p_1)$, (p_3, a_2), and (b_1, p_5) that $S_i \subset V_i$ and $S_i \cap F(V_i) = \emptyset$, $i = 1, 2, 3$.

We assume that there exist such natural ω_i, $i = 1, 2, 3$, that the contractions $F^{\omega_i}\big|_{U_i}$ have the following form

$$F^{\omega_1}\Big|_{U_1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_3 + (x - p_2) \\ a_2 + y + f_1(x - p_2, y) \end{pmatrix}, \tag{8}$$

$$F^{\omega_2}\Big|_{U_2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 + (x - a_1) + f_2(x - a_1, y - p_4) \\ p_5 + (y - p_4) \end{pmatrix}, \tag{9}$$

$$F^{\omega_3}\Big|_{U_3}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - b_2 + f_3(x - p_6, y - b_2) \\ p_1 - (x - p_6) \end{pmatrix}, \tag{10}$$

where f_i, $i = 1, 2, 3$, are the functions of the class C^r defined in the neighborhood of the origin of coordinates. We assume that

$$f_i(0,0) = 0, \quad \frac{\partial f_i(0,0)}{\partial x} = \frac{\partial f_i(0,0)}{\partial y} = 0, \quad i = 1,2,3. \tag{11}$$

We think that the inclusions $F^{\omega_i}(U_i) \subset S_i$, $i = 1, 2, 3$, are true. We assume that the sets U_i, $F(U_i)$, ..., $F^{\omega_i}(U_i)$, where $i = 1, 2, 3$, do not mutually intersect and no one of sets $F(U_i)$, ..., $F^{\omega_i - 1}(U_i)$, $i = 1, 2, 3$, intersects with any one of sets $V_1 \cup F(V_1)$, $V_2 \cup F(V_2)$, and $V_3 \cup F(V_3)$.

It follows from conditions (8)–(11) that the diffeomorphism F has a heteroclinic contour consisting of fixed points $O = (0, 0)$, $A = (a_1, a_2)$, and $B = (b_1, b_2)$ and trajectories of heteroclinic points $(0, p_1)$, (p_3, a_2), and (b_1, p_5); in all heteroclinic points stable and unstable manifolds intersect non-transversally. The functions f_i, $i = 1, 2, 3$, determine the character of tangency of stable and unstable manifolds in these points.

The points (p_3, a_2), (b_1, p_5), and $(0, p_1)$ are referred to as the points of finite-order tangency if natural l_1, l_2, and l_3 exist such that the relations are valid

$$\frac{\partial f_1(0,0)}{\partial x} = \ldots = \frac{\partial^{l_1 - 1} f_1(0,0)}{\partial x^{l_1 - 1}} = 0, \quad \frac{\partial^{l_1} f_1(0,0)}{\partial x^{l_1}} \neq 0,$$

$$\frac{\partial f_2(0,0)}{\partial y} = \ldots = \frac{\partial^{l_2 - 1} f_2(0,0)}{\partial y^{l_2 - 1}} = 0, \quad \frac{\partial^{l_2} f_2(0,0)}{\partial y^{l_2}} \neq 0,$$

$$\frac{\partial f_3(0,0)}{\partial x} = \ldots = \frac{\partial^{l_3 - 1} f_3(0,0)}{\partial x^{l_3 - 1}} = 0, \quad \frac{\partial^{l_3} f_3(0,0)}{\partial x^{l_3}} \neq 0.$$

The diffeomorphisms with non-transversal heteroclinic contours were studied in [3–5]. It was assumed that either the stable and unstable manifolds intersect transversally in all heteroclinic points or these points are the points of finite-order tangency.

To determine the characters of tangency of stable and unstable manifolds in heteroclinic points, we introduce the following objects from which we may determine the properties of the functions f_i, $i = 1, 2, 3$, below. Suppose Δ_k is a positive sequence such that $\lim\limits_{k \to +\infty} \Delta_k = 0$ and j_k is an increasing sequence of natural numbers. Suppose that a number δ exists satisfying the inequality $\max[\mu_3^{-1}, \lambda_2] < \delta < 1$ that

$$\Delta_k - \delta^{j_k} - \delta^{j_{k+1}} - \Delta_{k+1} > 0 \tag{12}$$

for any k.

Suppose that $U_i^-(k)$ and $U_i^+(k)i$, $i = 1, 2, 3$ are the following sequences of non-intersecting sets

$$U_1^-(k) = \left\{ \begin{array}{c} |x - (p_2 + \Delta_k)| \le 4\mu_3^{-\alpha j_k}, \\ \left|y - \lambda_1^{j_k}(p_1 - \Delta_k)\right| \le 2^{-1}(\lambda_1\lambda_2)^{j_k} \end{array} \right\},$$

$$U_1^+(k) = \left\{ \begin{array}{c} |x - (p_2 - \Delta_k)| \le \mu_3^{-j_k}, \\ \left|y - \lambda_1^{j_k}(p_1 + \Delta_k)\right| \le 8(\lambda_1\lambda_2)^{j_k} \end{array} \right\},$$

$$U_2^-(k) = \left\{ \begin{array}{c} \left|x - (a_1 + \lambda_2^{j_k}(p_3 + \Delta_k - a_1))\right| \le 2(\lambda_2\mu_3^{-\alpha})^{j_k}, \\ |y - (p_4 + \Delta_k)| \le 4^{-1}(\lambda_1\lambda_2\mu_2)^{j_k} \end{array} \right\},$$

$$U_2^+(k) = \left\{ \begin{array}{c} \left|x - (a_1 + \lambda_2^{j_k}(p_3 - \Delta_k - a_1))\right| \le 2(\lambda_2\mu_3^{-1})^{j_k}, \\ |y - (p_4 - \Delta_k)| \le 16(\lambda_1\lambda_2\mu_2)^{j_k} \end{array} \right\},$$

$$U_3^-(k) = \left\{ \begin{array}{c} |x - (p_6 + \Delta_k)| \le \lambda_2^{j_k}, \\ \left|y - (b_2 + \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2)\right| \le 8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k} \end{array} \right\},$$

$$U_3^+(k) = \left\{ \begin{array}{c} |x - (p_6 - \Delta_k)| \le 4\lambda_2^{j_k}, \\ \left|y - (b_2 + \lambda_3^{j_k}(p_5 - \Delta_k - b_2)\right| \le 32(\lambda_1\lambda_2\lambda_3\mu_2)^{j_k} \end{array} \right\}.$$

We think that for all k the inclusions $U_i^-(k) \subset U_i$ and $U_i^+(k) \subset U_i$, $i = 1, 2, 3$, are true.

We determine the sequences $\tau_i^-(k)$ and $\tau_i^+(k)$, $i = 1, 2, 3$, by the equalities

$$f_1(\Delta_k, \lambda_1^{j_k}(p_1 - \Delta_k)) = \mu_2^{-j_k}(p_4 + \Delta_k - a_2) - \lambda_1^{j_k}(p_1 - \Delta_k) + \tau_1^-(k),$$

$$f_1(-\Delta_k, \lambda_1^{j_k}(p_1 + \Delta_k)) = \mu_2^{-j_k}(p_4 - \Delta_k - a_2) - \lambda_1^{j_k}(p_1 + \Delta_k) + \tau_1^+(k),$$

$$f_2(\lambda_2^{j_k}(p_3 + \Delta_k - a_1), \Delta_k) = \mu_3^{-\alpha j_k}(p_6 + \Delta_k - b_1) - \lambda_2^{j_k}(p_3 + \Delta_k - a_1) + \tau_2^-(k),$$

$$f_2(\lambda_2^{j_k}(p_3 - \Delta_k - a_1), -\Delta_k) = \mu_3^{-j_k}(p_6 - \Delta_k - b_1) - \lambda_2^{j_k}(p_3 - \Delta_k - a_1) + \tau_2^+(k),$$

$$f_3(\Delta_k, \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2)) = \mu_1^{-j_k}(p_2 + \Delta_k) - \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2) + \tau_3^-(k),$$

$$f_3(-\Delta_k, \lambda_3^{j_k}(p_5 - \Delta_k - b_2)) = \mu_1^{-j_k}(p_2 - \Delta_k) - \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + \tau_3^+(k).$$

We assume that the functions f_i, $i = 1, 2, 3$, are such that for all k the following inequalities are fulfilled

$$\left|\tau_1^-(k)\right| < 8^{-1}(\lambda_1\lambda_2)^{j_k}, \quad \left|\tau_1^+(k)\right| < (\lambda_1\lambda_2)^{j_k},$$

$$\left|\tau_2^-(k)\right| < 2^{-1}(\lambda_2\mu_3^{-\alpha})^{j_k}, \quad \left|\tau_2^+(k)\right| < 2^{-1}(\lambda_2\mu_3^{-1})^{j_k}, \tag{13}$$

$$\left|\tau_3^-(k)\right| < (32)^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k}, \quad \left|\tau_3^+(k)\right| < (\lambda_1\lambda_2\lambda_3\mu_2)^{j_k}.$$

We assume that the derivatives of the functions f_i, $i = 1, 2, 3$, are such that

$$\left|\frac{\partial f_1(x,y)}{\partial x}\right| < (\mu_1\mu_2\mu_3^{\alpha})^{-j_k},$$

$$\left|\frac{\partial f_2(x,y)}{\partial y}\right| < (\mu_1\mu_2\mu_3^{\alpha})^{-j_k}, \tag{14}$$

$$\left|\frac{\partial f_3(x,y)}{\partial x}\right| < (\mu_1\mu_2\mu_3^{\alpha})^{-j_k}$$

for any k and any $(x, y) \in U_i^-(k)$, $(x, y) \in U_i^+(k)$, $i = 1, 2, 3$.

Conditions (13) mean that sequences Δ_k and j_k exist such that the values of the functions f_i, $i = 1, 2, 3$, are sufficiently close to the mentioned values in the corresponding points, and it follows from conditions (14) that the derivatives of the functions f_1 and f_3 with respect to the variable x and the derivative of the function f_2 with respect to the variable y are close to zero on the separated sets. From these conditions it follows that the tangencies of stable and unstable manifolds in the points $(0, p_1)$, (p_3, a_2), and (b_1, p_5) are not finite-order tangencies.

Theorem. *Suppose F is a diffeomorphism of a plane in itself of the class* C^r $(1 \le r \le \infty)$ *with non-transversal heteroclinic contour. Suppose that conditions* (2)–(14) *are fulfilled. Then, infinite sets of stable and completely unstable periodic points with characteristic exponents separated from zero lie in an arbitrary neighborhood of the point* $(p_2, 0)$.

Conditions (13) and (14) imply that for $1 \le r < \infty$ the following relation is true

$$\lim_{k\to+\infty} (\Delta_k)^r \zeta^{j_k} = +\infty,$$

where $\zeta = \max[\mu_1, \mu_2, (\mu_3)^\alpha, (\lambda_1)^{-1}, (\lambda_2)^{-1}, (\lambda_3)^{-\alpha}]$. At $r = \infty$ it is true for any natural l that

$$\lim_{k\to+\infty} (\Delta_k)^l \zeta^{j_k} = +\infty.$$

3. AUXILIARY LEMMAS

By $S_i^+(k)$ and $S_i^-(k)$, $i = 1, 2, 3$, we denote the following sequences of sets

$$S_1^-(k) = \left\{ \begin{matrix} \left| x - \mu_1^{-j_k}(p_2 + \Delta_k) \right| \le 4(\mu_1\mu_3^\alpha)^{-j_k} \\ |y - (p_1 - \Delta_k)| \le 2^{-1}\lambda_2^{j_k} \end{matrix} \right\},$$

$$S_1^+(k) = \left\{ \begin{matrix} \left| x - \mu_1^{-j_k}(p_2 - \Delta_k) \right| \le (\mu_1\mu_3)^{-j_k} \\ |y - (p_1 + \Delta_k)| \le 8\lambda_2^{j_k} \end{matrix} \right\},$$

$$S_2^-(k) = \left\{ \begin{matrix} |x - (p_3 + \Delta_k)| \le 2\mu_3^{-\alpha j_k} \\ \left| y - (a_2 + \mu_2^{-j_k}(p_4 + \Delta_k - a_2)) \right| \le 4^{-1}(\lambda_1\lambda_2)^{j_k} \end{matrix} \right\},$$

$$S_2^+(k) = \left\{ \begin{matrix} |x - (p_3 - \Delta_k)| \le 2\mu_3^{-\alpha j_k} \\ \left| y - (a_2 + \mu_2^{-j_k}(p_4 - \Delta_k - a_2)) \right| \le 16(\lambda_1\lambda_2)^{j_k} \end{matrix} \right\},$$

$$S_3^-(k) = \left\{ \begin{matrix} \left| x - (b_1 + \mu_3^{-\alpha j_k}(p_6 + \Delta_k - b_1) \right| \le (\lambda_2\mu_3^{-\alpha})^{j_k} \\ |y - (p_5 + \Delta_k)| \le 8^{-1}(\lambda_1\lambda_2\mu_2)^{j_k} \end{matrix} \right\},$$

$$S_3^+(k) = \left\{ \begin{matrix} \left| x - (b_1 + \mu_3^{-j_k}(p_6 - \Delta_k - b_1) \right| \le 4(\lambda_2\mu_3^{-1})^{j_k} \\ |y - (p_5 - \Delta_k)| \le 32(\lambda_1\lambda_2\mu_2)^{j_k} \end{matrix} \right\}.$$

We think that for any k the inclusions $S_i^+(k) \subset S_i$ and $S_i^-(k) \subset S_i$, $i = 1, 2, 3$, are valid.

The following equalities are obvious:

$$F^{j_k}(S_i^+(k)) = U_i^+(k), \quad i = 1, 2, 3,$$

$$F^{j_k}(S_i^-(k)) = U_i^-(k), \quad i = 1, 2,$$

$$F^{\alpha j_k}(S_3^-(k)) = U_3^-(k).$$

Lemma 1. *Suppose that the conditions of the theorem are fulfilled; then, the inclusions are true*

$$F^{3j_k+\omega_1+\omega_2+\omega_3}(S_1^+(k)) \subset S_1^+(k). \tag{15}$$

Proof. To prove the lemma, it is sufficient to prove the inclusions $F^{\omega_1}(U_1^+(k)) \subset S_2^+(k)$, $F^{\omega_2}(U_2^+(k)) \subset S_3^+(k)$, and $F^{\omega_3}(U_3^+(k)) \subset S_1^+(k)$. We fix k and prove the third of these inclusions; the remaining inclusions are proved analogously.

Suppose that $(x, y) \in U_3^+(k)$; then,

$$x = p_6 - \Delta_k + u, \quad |u| \le 4\lambda_2^{j_k},$$

$$y = b_2 + \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + v, \quad |v| \le 32(\lambda_1\lambda_2\lambda_3\mu_2)^{j_k}.$$

Suppose

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = F^{\omega_3}\begin{pmatrix}x\\ y\end{pmatrix};$$

then, we represent $\bar{x} = \mu_1^{-j_k}(p_2 - \Delta_k) + \bar{u}$ and $\bar{y} = p_1 + \Delta_k + \bar{v}$.

It follows from conditions (10) and the definition of $\tau_3^+(k)$ that

$$\mu_1^{-j_k}(p_2 - \Delta_k) + \bar{u} = \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + v + f_3(-\Delta_k + u, \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + v),$$

$$\bar{u} = v + \tau_3^+(k) + f_3(-\Delta_k + u, \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + v) - f_3(-\Delta_k, \lambda_3^{j_k}(p_5 - \Delta_k - b_2)),$$

$$\bar{v} = -u.$$

The mean value theorem implies that

$$\bar{u} = v + \tau_3^+(k) + \frac{\partial f_3(-\Delta_k + \theta_1 u, \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + \theta_2 v)}{\partial x}u + \frac{\partial f_3(-\Delta_k + \theta_1 u, \lambda_3^{j_k}(p_5 - \Delta_k - b_2) + \theta_2 v)}{\partial y}v,$$

where $0 < \theta_i < 1$, $i = 1, 2$.

From conditions (7), (11), and (14) we have

$$\left|\frac{\partial f_3(x, y)}{\partial y}\right| \le 2^{-1}, \quad (x, y) \in U_3^+(k),$$

$$\left|\frac{\partial f_3(x, y)}{\partial x}u\right| \le 4(\mu_1\mu_2\mu_3^{\alpha})^{-j_k}\lambda_2^{j_k} < (\lambda_1\lambda_2\lambda_3\mu_2)^{j_k}.$$

We obtain

$$|\bar{u}| < 50(\lambda_1\lambda_2\lambda_3\mu_2)^{j_k} < (\mu_1\mu_2)^{-j_k},$$

$$|\bar{v}| = |u| \le 4\lambda_2^{j_k} < 8\lambda_2^{j_k}.$$

The lemma is proved.

Lemma 2. *Suppose that the conditions of the theorem are satisfied; then, the inclusions are true*

$$S_1^-(k) \subset F^{(2+\alpha j_k)+\omega_1+\omega_2+\omega_3}(S_1^-(k)). \tag{16}$$

Proof. To prove the lemma, it is sufficient to prove the inclusions $S_2^-(k) \subset F^{\omega_1}(U_1^-(k))$, $S_3^-(k) \subset F^{\omega_2}(U_2^-(k))$, and $S_1^-(k) \subset F^{\omega_3}(U_3^-(k))$. We fix k and prove the third of these inclusions; the remaining inclusions are proved analogously.

Suppose $(x, y) \in U_3^-(k)$; then,

$$x = p_6 + \Delta_k + u, \quad |u| \le \lambda_2^{j_k},$$

$$y = b_2 + \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2) + v, \quad |v| \le 8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k}.$$

We set

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = F^{\omega_3}\begin{pmatrix}x\\ y\end{pmatrix}$$

and represent

$$\bar{x} = \mu_1^{-j_k}(p_2 + \Delta_k) + \bar{u}, \quad \bar{y} = p_1 - \Delta_k + \bar{v},$$

where

$$\bar{u} = v + \tau_3^-(k) + f_3(\Delta_k + u, \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2) + v) - f_3(\Delta_k, \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2)),$$
$$\bar{v} = -u.$$

For any fixed u the function $\bar{u}(v)$ is differentiable and

$$\frac{d\bar{u}}{dv} > 0;$$

consequently, this function is monotonic and $\bar{u}(v) \in [\bar{u}(-8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k}), \bar{u}(8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k})]$.

Using the mean value theorem, we write

$$\bar{u} = v + \tau_3^-(k) + \frac{\partial f_3(\Delta_k + \theta_1 u, \lambda_3^{\alpha j_k}(p_5 - \Delta_k - b_2) + \theta_2 v)}{\partial x} u$$
$$+ \frac{\partial f_3(\Delta_k + \theta_1 u, \lambda_3^{\alpha j_k}(p_5 + \Delta_k - b_2) + \theta_2 v)}{\partial y} v,$$

where $0 < \theta_i < 1$, $i = 1, 2$.

From conditions (7), (11), and (14) we obtain

$$\left|\frac{\partial f_3(x, y)}{\partial y}\right| \le 8^{-1}, \quad (x, y) \in U_3^-(k),$$

$$\left|\frac{\partial f_3(x, y)}{\partial x} u\right| \le (\mu_1\mu_2\mu_3^{\alpha})^{-j_k}\lambda_2^{j_k} < (64)^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k},$$

$$|\bar{u} - v| < (16)^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k};$$

consequently,

$$\bar{u}(8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k}) > (16)^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k} \ge 4(\mu_1\mu_3^{\alpha})^{-j_k},$$
$$\bar{u}(-8^{-1}(\lambda_1\lambda_2\lambda_3^{\alpha}\mu_2)^{j_k}) < -4(\mu_1\mu_3^{\alpha})^{-j_k}.$$

It is clear that $|\bar{v}| = |u|$. Inclusions (16) follow from the last equalities and inequalities.

The lemma is proved.

4. PROOF OF THE THEOREM

It follows from conditions (15) and (16) that for any k the periodic points of diffeomorphism lie in the sets $U_1^-(k)$ and $U_1^+(k)$. Suppose that $u^+(k) \in U_1^+(k)$ and $u^-(k) \in U_1^-(k)$ are periodic points; it is clear that $F^{3j_k+\omega_1+\omega_2+\omega_3}(u^+(k)) = u^+(k)$ and $F^{(2+\alpha)j_k+\omega_1+\omega_2+\omega_3}(u^-(k)) = u^-(k)$. Let

$$\Phi_k^+ = DF^{3j_k+\omega_1+\omega_2+\omega_3}(u^+(k)),$$
$$\Phi_k^- = DF^{(2+\alpha)j_k+\omega_1+\omega_2+\omega_3}(u^-(k)).$$

From conditions (2)–(4), (8)–(10), (11), and (14) it follows that these matrices may be represented as

$$\Phi_k^+ = \Psi_3^+(k)\Psi_2^+(k)\Psi_1^+(k),$$
$$\Phi_k^- = \Psi_3^-(k)\Psi_2^-(k)\Psi_1^-(k),$$

where

$$\Psi_1^+(k) = \begin{pmatrix} \lambda_2^{j_k} & 0 \\ \varphi_1^+(k)\mu_2^{j_k} & (1 + \psi_1^+(k))\mu_2^{j_k} \end{pmatrix},$$

$$\Psi_2^+(k) = \begin{pmatrix} (1+\psi_2^+(k))\mu_3^{j_k} & \varphi_2^+(k)\mu_3^{j_k} \\ 0 & \lambda_3^{j_k} \end{pmatrix},$$

$$\Psi_3^+(k) = \begin{pmatrix} \varphi_3^+(k)\mu_1^{j_k} & (1+\psi_3^+(k))\mu_1^{j_k} \\ -\lambda_1^{j_k} & 0 \end{pmatrix},$$

$$\Psi_1^-(k) = \begin{pmatrix} \lambda_2^{j_k} & 0 \\ \varphi_1^-(k)\mu_2^{j_k} & (1+\psi_1^-(k))\mu_2^{j_k} \end{pmatrix},$$

$$\Psi_2^-(k) = \begin{pmatrix} (1+\psi_2^-(k))\mu_3^{\alpha j_k} & \varphi_2^-(k)\mu_3^{\alpha j_k} \\ 0 & \lambda_3^{\alpha j_k} \end{pmatrix},$$

$$\Psi_3^-(k) = \begin{pmatrix} \varphi_3^-(k)\mu_1^{j_k} & (1+\psi_3^-(k))\mu_1^{j_k} \\ -\lambda_1^{j_k} & 0 \end{pmatrix},$$

where

$$\lim_{k\to+\infty} \psi_i^+(k) = 0, \quad \lim_{k\to+\infty} \psi_i^-(k) = 0,$$

$$\left|\varphi_i^-(k)\right| < (\mu_1\mu_2\mu_3^{\alpha})^{-j_k},$$

$$\left|\varphi_i^+(k)\right| < (\mu_1\mu_2\mu_3^{\alpha})^{-j_k}, \quad i = 1, 2, 3.$$

Suppose that $\beta_1 = \lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3$ and $\beta_2 = \lambda_1\mu_1\lambda_2\mu_2(\lambda_3\mu_3)^{\alpha}$; then, the determinants of the matrices $\det\Phi_k^+$ and $\det\Phi_k^-$ have the form

$$\det\Phi_k^+ = \beta_1^{j_k}(1+h_1(k)),$$

$$\det\Phi_k^- = \beta_2^{j_k}(1+h_2(k)),$$

where $\lim_{k\to+\infty} h_i(k) = 0$, $i = 1, 2$.

The direct computations of the traces of the matrices Φ_k^+ and Φ_k^- show that

$$\begin{aligned}\operatorname{Tr}\Phi_k^+ &= (\lambda_2\mu_1\mu_3)^{j_k}\varphi_3^+(k)(1+\psi_2^+(k)) + (\lambda_3\mu_1\mu_2)^{j_k}\varphi_1^+(k)(1+\psi_3^+(k)) \\ &- (\lambda_1\mu_2\mu_3)^{j_k}\varphi_2^+(k)(1+\psi_1^+(k)) + (\mu_1\mu_2\mu_3)^{j_k}\varphi_1^+(k)\varphi_2^+(k)\varphi_3^+(k),\end{aligned}$$

$$\begin{aligned}\operatorname{Tr}\Phi_k^- &= (\lambda_2\mu_1(\mu_3)^{\alpha})^{j_k}\varphi_3^-(k)(1+\psi_2^-(k)) + ((\lambda_3)^{\alpha}\mu_1\mu_2)^{j_k}\varphi_1^-(k)(1+\psi_3^-(k)) \\ &- (\lambda_1\mu_2(\mu_3)^{\alpha})^{j_k}\varphi_2^-(k)(1+\psi_1^-(k)) + (\mu_1\mu_2(\mu_3)^{\alpha})^{j_k}\varphi_1^-(k)\varphi_2^-(k)\varphi_3^-(k).\end{aligned}$$

Suppose

$$\gamma_1 = \max[\lambda_1(\mu_1)^{-1}(\mu_3)^{-\alpha+1}, \lambda_2(\mu_2)^{-1}(\mu_3)^{-\alpha+1}, \lambda_3(\mu_3)^{-\alpha}, (\mu_3)^{-\alpha+1}(\mu_1\mu_2\mu_3^{\alpha})^{-2}],$$

$$\gamma_2 = \max[\lambda_1(\mu_1)^{-1}, \lambda_2(\mu_2)^{-1}, (\lambda_3)^{\alpha}(\mu_3)^{-\alpha}, (\mu_1\mu_2\mu_3^{\alpha})^{-2}].$$

It is clear that

$$\gamma_1 < \beta_1 < 1, \quad \gamma_2 < 1 < \beta_2. \tag{17}$$

The traces of the matrices Φ_k^+ and Φ_k^- may be represented as

$$\operatorname{Tr}\Phi_k^+ = \gamma_1^{j_k} q_k^+,$$

$$\operatorname{Tr}\Phi_k^- = \gamma_2^{j_k} q_k^-,$$

where q_k^+ and q_k^- are bounded values.

Suppose that $\rho_i^+(k)$ and $\rho_i^-(k)$, $i = 1, 2$, are the eigenvalues of the matrices Φ_k^+ and Φ_k^-, respectively. Due to inequalities (17) we have

$$\left|\rho_i^+(k)\right| = (\det \Phi_i^+(k))^{0.5},$$
$$\left|\rho_i^-(k)\right| = (\det \Phi_i^-(k))^{0.5}, \quad i = 1, 2.$$

Suppose $\nu_i^+(k)$ and $\nu_i^-(k)$, $i = 1, 2$, are the characteristic exponents of the periodic points $u^+(k)$ and $u^-(k)$. Then, by formula (1) we have

$$\nu_i^+(k) = (3j_k + \omega_1 + \omega_2 + \omega_3)^{-1} \ln\left|\rho_i^+(k)\right|,$$

$$\nu_i^-(k) = ((2 + \alpha)j_k + \omega_1 + \omega_2 + \omega_3)^{-1} \ln\left|\rho_i^-(k)\right|;$$

therefore,

$$\lim_{k\to+\infty} \nu_i^+(k) = 6^{-1}\ln(\beta_1) < 0, \quad i = 1, 2,$$

$$\lim_{k\to+\infty} \nu_i^-(k) = (4 + 2\alpha)^{-1}\ln(\beta_2) > 0, \quad i = 1, 2.$$

The latter inequalities prove the theorem.

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