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Dedicated to Vasiliy Mikhailovich Babich

SCATTERING OF A SURFACE WAVE IN A POLYGONAL DOMAIN WITH IMPEDANCE BOUNDARY

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The two-dimensional (2D) domain under study is bounded from below by two semi-infinite and, between them, two finite straight lines; on each of the straight lines (segments), a usually different impedance boundary condition is imposed. An incident surface wave, propagating from infinity along one semi-infinite segment of the polygonal domain, excites outgoing surface waves both on the same segment (a reflected wave) and on the second semi-infinite segment (a transmitted wave); in addition, a circular (cylindrical) outgoing wave will be generated in the far field. The scattered wave field satisfies the Helmholtz equation and the Robin (in other words, impedance) boundary conditions as well as some special integral form of the Sommerfeld radiation conditions. It is shown that a classical solution of the problem is unique. By the use of some known extension of the Sommerfeld–Malyuzhinets technique, the problem is reduced to functional Malyuzhinets equations and then to a system of integral equations of the second kind with the integral operator depending on a characteristic parameter. The Fredholm property of the equations is established, which also leads to the existence of the solution for noncharacteristic values of the parameter. From the Sommerfeld integral representation of the solution, the far-field asymptotics is developed. Numerical results for the scattering diagram are also presented.

§1. Introduction

There are different methods to solve problems of diffraction in wedge-shaped or polygonal domains. Among them, there are approaches based on the Fourier

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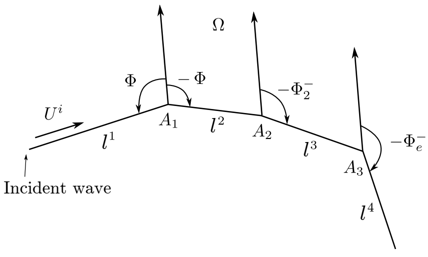

Figure 1. Diffraction of a surface wave in a polygonal domain.

transform and on single (double) layer potentials (see, e.g., [1–6], and the references therein), the Kontorovich–Lebedev transform and "spectral" integral equations (for penetrable wedges, see, e.g., [7, 8] and references therein). For diffraction by a cone, the Kontorovich–Lebedev transform was efficiently used in [9–13]. The Sommerfeld–Malyuzhinets (SM) technique, Malyuzhinets functional equations and their reduction to integral ones were discussed in [13–15]. Various useful references and discussions can be also found in [16].

Developments of the above mentioned and other specific techniques for complex wedges with polygonal boundaries are also discussed in the literature, see, e.g., [17]. An interesting and important development of the Sommerfeld–Malyuzhinets (SM) technique with applications to diffraction by an impedance polygon was given in [18]. In fact, basic ideas of this extension of the SM technique lie in the core of our study in the present work.

Two other approaches should also be mentioned: [19] (Lax pairs and simultaneous spectral analysis) and [20] (a probabilistic approach). Application and some extension of the Kontorovich–Lebedev technique, applied to diffraction by two shifted wedges, were considered in our paper [21].

Let the polygonal domain Ω in Figure 1 have the boundary $\overline{l^1 \cup l^2 \cup l^3 \cup l^4}$ with two semi-infinite segments $l^{1,4}$ of the boundary and two finite segments $l^{2,3}$ with lengths $d_{1,2}$ respectively. The angular points A_1, A_2, A_3 (vertices) are the respective ends of the segments. The incident surface wave $U^i(r,\varphi)$ propagates along the segment l^1 from infinity. It is a simple solution of the wave (Helmholtz) equation satisfying the Robin or, in other words, the impedance boundary condition[1]

$$\Big(\frac{\partial U^i}{\partial n} - \mathrm{i}k\eta_1\, U^i\Big)\Big|_{l^1} = 0,$$

[1]The dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ on time is assumed and omitted throughout the paper.

where $\frac{\partial}{\partial n} = \frac{1}{r}\frac{\partial}{\partial \varphi}$, n is the outward normal to Ω, k is the wave number. Next, (r, φ) are the polar coordinates attributed to the point A_1 such that the equation of l^1 is $\varphi = \Phi$ and $r > 0$, η_1 is the surface impedance independent of k, $\eta_1 = \sin\vartheta_+$ with $\Re(\vartheta_+) = 0$ and $\Im(\vartheta_+) < 0$. Under these assumptions the desired solution representing the surface wave is easily found:

$$U^i(r, \varphi) = \exp\{-\mathrm{i}kr\cos[\varphi - \varphi_s]\},$$

$\varphi_s = \Phi - \vartheta_+$. It is easily verified that this wave is bounded on the boundary l^1, $k > 0$.

On the other segments of the boundary similar impedance boundary conditions are valid with the surface impedances $\eta_2 = \sin\vartheta_1$, $\eta_3 = \sin\vartheta_2$, $\eta_4 = \sin\vartheta_-$; we assume that $\Re(\vartheta_-) = 0$ and $\Im(\vartheta_-) < 0$, which, in particular, implies that l^4 can also support propagating surface waves. Other restrictions on the geometry of the domain and on the impedances $\eta_{2,3}$ will be fixed below.

The surface wave U^i is incident along l^1 from infinity, then the scattered wave field consists of a reflected surface wave along l^1 and a transmitted surface wave along l^4, both outgoing to infinity. At the same time a circular (or cylindrical in 3D) wave propagates to infinity. The excitation coefficients of the surface waves as well as the diffraction coefficient of the circular wave are the most important characteristics of the wave scattering. Their study is a main goal in this work. These characteristics cannot be found from any local considerations, for instance, as in the case of the reflection coefficients for a plane impedance boundary and, hence, require a complete solution of the scattering problem in question. We solve the problem, compute the far-field asymptotics, and give representations for these characteristics of scattering in terms of the solution of a matrix integral equation. We study the Fredholm property, the uniqueness and the solvability of this equation. Remark that we claim but not actually prove the existence of the solution for the problem under study, which requires some additional technical work, which goes beyond the scope of this paper. We also aim at specifying the scope of applicability of the approach in [18] considering the problem of scattering of a surface wave in its rigorous formulation. In particular, this requires a new form of the radiation condition that allows for taking into account the outgoing surface waves. On the other hand, we expect to find some limitations on the geometry of the polygonal domain and on the surface impedances. Our further goal, not dealt with in this work, is to compare the efficiency of the extended SM technique with other methods like Geometrical Theory of Diffraction or its uniform versions.

In the next section we formulate the problem, postulating an appropriate radiation and Meixner's conditions, and study the uniqueness of the classical solution. From a physical point of view, uniqueness is based on the assumption that one of the finite segments of the boundary absorbs energy. This circumstance requires the positivity of the real part of the surface impedance on this segment, see Subsection 2.5 in [15]. The other segments of the boundary are reactive (neither active nor absorbing), i.e., the real parts of their impedances are zero. Remark that the impedances are independent of the wave number. In the third section, by means of the Sommerfeld integral representation of the wave field, we derive the problem for the functional (Malyuzhinets) equations in terms of the Sommerfeld transformants in the framework of the mentioned extension of the SM technique. In the fourth section, we reduce the problem for the functional equations on the complex plane to a matrix integral equation of the second kind and give integral representations of the meromorphic transformants in terms of solution of the integral equation. In the fifth section we study the Fredholm property of the integral equation in question, making use of the so-called analytic Fredholm alternative. The last-mentioned two sections contain the most cumbersome derivations and estimates. In the sixth section appropriate singularities of the Sommerfeld transformants are found and the far-field asymptotics are developed. In this way, the expressions for the excitation or diffraction coefficients are obtained.

Section 7 deals with numerical calculation of the far field which is based on our analytical results in this work. Section 8 summarizes the main features of the present paper. Finally, Appendix is devoted to the derivation of the Malyuzhinets functional equations under consideration.

§2. Formulation

In the domain Ω (Figure 1), the total wave field (if kr is sufficiently large, $\Phi - \epsilon_1 < \varphi \leqslant \Phi$, k is fixed)

$$u = U^i + u^{sc} \tag{1}$$

is represented by the sum of the incident surface wave U^i propagating along l^1 and the unknown scattered field $u^{sc}(r, \varphi)$ written in the polar coordinates (r, φ) attributed to the vertex A_1 of the boundary. The representation (1) is valid and physically motivated for the observation angles φ such that $\Phi - \epsilon_1 < \varphi \leqslant \Phi$ for some small $\epsilon_1 > 0$. We formulate the problem under consideration as a scattering problem. In our case, this means that we prescibe one of the terms in the far-field asymptotics explicitly assuming that, after solving the problem, the solution must have the prescibed term in the asymptotics together with other summands. In our case, this term is U^i that represents the incident surface wave, which is extracted in the asymptotics only in some angular neighborhood of the side l^1 of the boundary. Some notation pertaning to the scatterer are given in Figure 1.

The total field satisfies the Helmholtz equation in Ω:

$$(\triangle + k^2)u = 0 \tag{2}$$

and the impedance boundary conditions on the segments of the boundary

$$\Big(\frac{\partial}{\partial n} - \mathrm{i}k\eta_j\Big)u|_{l^j} = 0, \quad j = 1,2,3,4, \tag{3}$$

where n is the outward normal to Ω on the boundary $\partial\Omega$, $\eta_j \quad (j = 1,2,3,4)$ are the surface impedances. It is convenient to introduce the parameters $\vartheta_+, \vartheta_-, \vartheta_1$, and ϑ_2 as follows:

$$\eta_1 = \sin\vartheta_+, \quad \eta_4 = \sin\vartheta_-, \quad \eta_2 = \sin\vartheta_1, \quad \eta_3 = \sin\vartheta_2, \tag{4}$$

which are independent of the wave number k. The following limitations for the impedances are assumed

$$\pi/2 \geqslant \Re(\vartheta_1) > 0, \quad \Re(\vartheta_2) = 0,^2 \quad \Re(\vartheta_\pm) = 0, \quad \Im(\vartheta_+) < 0. \tag{5}$$

The restriction for the real part of ϑ_1 means absorption of the wave field energy on the segment l^2 of the boundary. The inequality $\Im(\vartheta_+) < 0$ ensures the existence of propagating surface waves along the segment l^1 of the boundary. The same is valid for the imaginary part of the impedance of the segment l^4, however, the approach is also applicable to the case of $\Im(\vartheta_-) \geqslant 0$.

The lengths of the finite segments are d_1, d_2. It is also implied that (see Figure 1)

$$\Phi > \pi/2, \quad \Phi_2^- > \Phi, \quad \Phi_e^- > \Phi_2^-, \quad \text{and} \quad \Phi + \Phi_e^- \leqslant 2\pi. \tag{6}$$

The limitations on the geometry of the polygonal domain Ω in (6) are of technical origin and they are explained in §4 dealing with the derivation of the integral equations. They actually mean that the polygonal scatterer $R^2 \setminus \Omega$ is convex. The approach can also be applied to nonconvex domains, however, the analysis should be modified appropriately.

Meixner's conditions at the angular points are satisfied:

$$\Im\Big(\int_{S_\epsilon^j} \frac{\partial u}{\partial n}\,\overline{u}\,\mathrm{d}s\Big) \to 0, \quad j = 1,2,3, \tag{7}$$

as $\epsilon \to 0$, where S_ϵ^j is a part of a circumference of radius ε in Ω centered at the jth vertex, i.e., the energy flux through S_ϵ^j vanishes in the limit. Another equivalent form of Meixner's conditions (see the discussion in Chapter 1 of [15]) is also of use herein

$$u = C^j + O(\rho_j^{\delta_j}), \quad \text{as} \quad \rho_j \to 0,$$

[2]For some technical reason we shall additionally assume that $\Re(\vartheta_2) > 0$ although this assumption is not necessary for uniqueness.

where $\delta_j > 0$, $j = 1, 2, 3$ depend on the openings of the angles with the vertices A_j, respectively, ρ_j is the distance from A_j in Ω.

Finally, the radiation condition at infinity reads

$$\begin{aligned}\int\limits_{S_R^i} \left|\frac{\partial u}{\partial \rho} - \mathrm{i}ku\right|^2 \mathrm{d}s + \int\limits_{S_{R,b}^-} \left|\frac{\partial u}{\partial \rho} - \mathrm{i}k\cos\vartheta_- u\right|^2 \mathrm{d}s \\ + \int\limits_{S_{R,b}^+} \left|\frac{\partial u^{sc}}{\partial \rho} - \mathrm{i}k\cos\vartheta_+ u^{sc}\right|^2 \mathrm{d}s \to 0,\end{aligned} \tag{8}$$

as $R \to \infty$ and $S_R^i = S_R \backslash (S_{R,b}^+ \cup S_{R,b}^-)$, where S_R is the part of the circumference in Ω having the radius $\rho = R$ and centered at the origin O (Figure 2),

$$S_{R,b}^{\pm} = \{(\rho, \psi)\colon\ \rho = R,\ 0 < \Psi \mp \psi < R^{-1+\varkappa}\},$$

$\varkappa > 0$ is small. In the definition of the arcs $S_{R,b}^{\pm}$ we make use of the polar coordinates (ρ, ψ) in the angle contained in Ω (see Figure 2) with the sides $\psi = \pm\Psi$ and with the vertex O at the point of intersection of the continuations of l^1 and l^4. (2Ψ is the opening of this angle, $2\Psi = \Phi + \Phi_e^-$.) These arcs correspond to some small angular neighborhoods of the semi-infinite segments l^1 and l^4 as $R \to \infty$.

The radiation condition in the Rellich integral form (8) is a generalization of the standard Sommerfeld radiation condition taking into account the outgoing surface waves along l^1 and l^4 (see also [13, Chapter 4]). It implies that for S_R^i, i.e., outside some close neighborhoods of l^1 and l^4, the scattered field has the following far-field asymptotic behavior in the leading approximation

$$u^{sc} \sim \frac{e^{\mathrm{i}k\rho + \mathrm{i}\pi/4}}{\sqrt{2\pi k\rho}} D_s(\psi) \left(1 + O\left(\frac{1}{k\rho}\right)\right), \tag{9}$$

$D_s(\,\cdot\,)$ is the unknown diffraction coefficient of the circular (cylindrical in 3D) wave, $k\rho = kR \to \infty$, whereas the reflected (u_s^+) and transmitted (u_s^-) surface waves are exponentially small for these directions corresponding to S_R^i. On the other hand, for the directions corresponding to $S_{R,b}^{\pm}$, i.e., in some close neighborhoods of l^1 and l^4, one has

$$u^{sc} \sim u_s^{\pm}(\rho, \psi) + \frac{e^{\mathrm{i}k\rho + \mathrm{i}\pi/4}}{\sqrt{2\pi k\rho}} D_s(\psi) \Big(1 + O\Big(\frac{1}{k\rho}\Big)\Big) \tag{10}$$

with $+$ for l^1 and $-$ for l^4 on the right-hand side of (10), where the

$$u_s^{\pm}(\rho, \psi) = c_s^{\pm} \exp\left\{\mathrm{i}k\rho\cos[\psi \mp \psi_s^{\pm}]\right\}$$

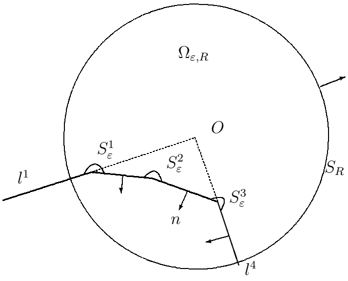

Figure 2. To the proof of uniqueness.

are the outgoing surface waves propagating along l^1 and l^4, respectively, with yet unknown excitation coefficients $c_s^{\pm}$, $\psi_s^{\pm} = \Psi + \vartheta_{\pm}$.

The analysis to follow will enable us, in particular, to determine the excitation and diffraction coefficients of the outgoing surface and circular waves (see §6 below), which are the main characteristics of scattering for the problem (1)–(8). We should emphasize that the diffraction coefficient D_s and the excitation coefficients $c^s_{\pm}$ parametrically depend on the wave number k, however, we omit this dependence throughout this paper. In this work we are looking for the *classical* solution of the problem at hand, which means that $u \in C^2_{\text{loc}}(\Omega)$ has continuous traces $\partial_n u|_{l^j}$ and $u|_{l^j}$, $j = 1,2,3,4$, and is such that (1)–(8) are satisfied.

2.1. Uniqueness of the classical solution. In this section, taking into account the assumptions above, we prove the following fact.

Theorem 2.1. *For $k > 0$ the classical solution of problem* (1)–(8), *if exists, is unique.*

Proof. We introduce the domain $\Omega_{\varepsilon,R}$ (Figure 2) with the boundary $\partial\Omega_{\varepsilon,R}$ consisting of the arc S_R and those S^j_ε, of small radius ε with center at A_j, $j = 1,2,3$ and of the finite segments $l^j_{\varepsilon,R}$, $j = 1,4$ (subsets of $l^{1,4}$), l^j_ε, $j = 2,3$ (subsets of $l^{2,3}$) with the end-points at the respective ends of the arcs.

In a traditional manner, from the Helmholtz equation and by use of Green's identity we arrive, taking the imaginary part, at

$$\Im\Bigg(\int\limits_{\Omega_{\varepsilon,R}} \Big(-|\nabla u|^2 + k^2|u|^2 \Big) \mathrm{d}V + \int\limits_{\partial\Omega_{\varepsilon,R}} \frac{\partial u}{\partial n}\, \bar{u}\, \mathrm{d}s \Bigg) = 0$$

or

$$\Im\Big(\int\limits_{\partial\Omega_{\varepsilon,R}} \frac{\partial u}{\partial n}\,\bar u\,\mathrm{d}s\Big) = 0. \tag{11}$$

From (11), making use of the boundary conditions, one has

$$\Im\Big(\mathrm{i}k\sum_{j=1}^{4}\eta_j \int\limits_{l^j_{\varepsilon,R}} |u|^2\,\mathrm{d}s + \sum_{j=1,2,3}\int\limits_{S^j_\varepsilon} \frac{\partial u}{\partial n}\,\bar u\ \mathrm{d}s + \int\limits_{S_R} \frac{\partial u}{\partial n}\,\bar u\ \mathrm{d}s\Big) = 0. \tag{12}$$

We let $\varepsilon \to 0$. Taking into account the conditions restricting the impedances (5), after simple transformation in (12) we arrive at

$$\Im\Big(\mathrm{i}k\Re(\eta_2)\int\limits_{l^2_R} |u|^2\,\mathrm{d}s + \int\limits_{S^i_R}\Big(\frac{\partial u}{\partial\rho}\Big)\,\bar u\ \mathrm{d}s + \sum_{\pm}\int\limits_{S^{\pm}_{b,R}}\Big(\frac{\partial u}{\partial\rho}\Big)\ \bar u\ \mathrm{d}s\Big) = 0$$

and

$$\begin{aligned} 0 \leqslant \Im\Big(\mathrm{i}k\Re(\eta_2)\int\limits_{l^2_R} |u|^2\,\mathrm{d}s\Big) &= -\Im\Big(\int\limits_{S^i_R}\Big(\frac{\partial u}{\partial\rho} - \mathrm{i}ku + \mathrm{i}ku\Big)\,\bar u\ \mathrm{d}s \\ &+ \sum_{\pm}\int\limits_{S^{\pm}_{b,R}}\Big(\frac{\partial u}{\partial\rho} - \mathrm{i}k\cos\vartheta_{\pm}u + \mathrm{i}k\cos\vartheta_{\pm}u\Big)\ \bar u\ \mathrm{d}s\Big). \end{aligned} \tag{13}$$

The right-hand side in (13) is nonnegative, thus we obtain

$$\begin{aligned} 0 &\leqslant k\int\limits_{S^i_R} |u|^2\,\mathrm{d}s + \sum_{\pm} k\cos\vartheta_{\pm}\int\limits_{S^{\pm}_{R,b}} |u|^2\,\mathrm{d}s \\ &= -\Im\Big(\int\limits_{S^i_R}\Big(\frac{\partial u}{\partial\rho} - \mathrm{i}ku\Big)\,\bar u\ \mathrm{d}s + \sum_{\pm}\int\limits_{S^{\pm}_{R,b}}\Big(\frac{\partial u}{\partial\rho} - \mathrm{i}k\cos\vartheta_{\pm}u\Big)\ \bar u\,\mathrm{d}s\Big) \\ &\leqslant \Big[\int\limits_{S^i_R}\Big|\frac{\partial u}{\partial\rho} - \mathrm{i}ku\Big|^2\ \mathrm{d}s\Big]^{1/2}\Big[\int\limits_{S^i_R} |u|^2\,\mathrm{d}s\Big]^{1/2} \\ &+ \sum_{\pm}\Big[\int\limits_{S^{\pm}_{R,b}}\Big|\frac{\partial u}{\partial\rho} - \mathrm{i}k\cos\vartheta_{\pm}u\Big|^2\ \mathrm{d}s\Big]^{1/2}\Big[\int\limits_{S^{\pm}_{R,b}} |u|^2\,\mathrm{d}s\Big]^{1/2}. \end{aligned} \tag{14}$$

It is now useful to introduce notation

$$H_R(u) = \int\limits_{S_R^i} |u|^2\, \mathrm{d}s + \sum_{\pm} \cos\vartheta_\pm \int\limits_{S_{R,b}^\pm} |u|^2\, \mathrm{d}s$$

We have that

$$\int\limits_{S_R^\pm} |u|^2\, \mathrm{d}s\, / H_R(u) \leqslant 1, \qquad \int\limits_{S_{b,R}^\pm} |u|^2\, \mathrm{d}s / H_R(u) \leqslant 1$$

then from (14) we find

$$\begin{aligned} 0 &\leqslant \lim_{R\to\infty} k\Bigg(\int\limits_{S_R^i} |u|^2\, \mathrm{d}s + \sum_{\pm} \cos\vartheta_\pm \int\limits_{S_{R,b}^\pm} |u|^2\, \mathrm{d}s \Bigg)^{1/2} \\ &\leqslant \lim_{R\to\infty} \Bigg(\Bigg[\int\limits_{S_R^i} \left| \frac{\partial u}{\partial \rho} - \mathrm{i}ku \right|^2 \mathrm{d}s \Bigg]^{1/2} + \sum_{\pm} \Bigg[\int\limits_{S_{R,b}^i} \left| \frac{\partial u}{\partial \rho} - \mathrm{i}k\cos\vartheta_\pm u \right|^2 \mathrm{d}s \Bigg]^{1/2} \Bigg). \end{aligned} \tag{15}$$

In view of the radiation condition (8), the right-hand side of (15) is zero and we arrive at

$$\lim_{R\to\infty} \Bigg(\int\limits_{S_R^i} |u|^2\, \mathrm{d}s + \sum_{\pm} \cos\vartheta_\pm \int\limits_{S_{R,b}^\pm} |u|^2\, \mathrm{d}s \Bigg) = 0$$

and, as a consequence of the identity in (14), we have

$$\lim_{R\to\infty} \Im\Bigg(\int\limits_{S_R^i} \left(\frac{\partial u}{\partial \rho} - \mathrm{i}ku \right) \bar{u}\, \mathrm{d}s + \sum_{\pm} \int\limits_{S_{R,b}^\pm} \left(\frac{\partial u}{\partial \rho} - \mathrm{i}k\cos\vartheta_\pm u \right) \bar{u}\, \mathrm{d}s \Bigg) = 0.$$

Now, letting $R \to \infty$ on the right-hand side of the identity in (13), we conclude that

$$\Im\Bigg(\mathrm{i}k\Re(\eta_2) \int\limits_{l^2} |u|^2 \mathrm{d}s \Bigg) = 0$$

and

$$u|_{l^2} = 0.$$

From the impedance boundary condition on l^2 we also have

$$\left.\frac{\partial u}{\partial n}\right|_{l^2} = 0.$$

From the last two formulas we would like to conclude that $u = 0$ in Ω. To this end, notice that we deal with the Helmholtz equation, which is elliptic in Ω. We then consider some symmetric (with respect to) l^2 neighborhood of this segment. We are able to continue u as an odd function from the part of such a neighborhood that belongs to Ω to the symmetric part. Then, as is known, the Helmholtz equation will be valid in the whole symmetric neighborhood of l^2. As a result, u is a solution of the elliptic equation in a domain (the symmetric neighborhood of l^2) and satisfies homogeneous Cauchy conditions on l^2. It is known that such a solution is trivial[3] ($u \equiv 0$) in the domain of ellipticity and hence in Ω. □

§3. A modified Sommerfeld–Malyuzhinets (SM) approach and the problem for the Malyuzhinets functional equations

In this section we make use of the known extension of the SM technique [18] (see also Appendix) and apply it in order to get Sommerfeld integral representations of the wave field in Ω and formulate a problem for the respective system of Malyuzhinets functional equations. The Sommerfeld integrals solve the Helmholtz equation, whereas applying Malyuzhinets' inversion theorem of the Sommerfeld integrals (see [22]) to the boundary conditions leads to the functional equations for the Sommerfeld transformants (i.e., unknown functions in the integrands). This is a traditional way for the SM approach.

We introduce the polar axis at A_1 as shown in Figure 1 and polar coordinates (r, φ), and assume that the polygonal boundary is outside the angle $r > 0$, $|\varphi| < \Phi$. The other two auxiliary polar coordinate systems (ρ_2, φ_2) and (ρ_e, φ_e) are attributed to A_2 and A_3, respectively. In the coordinates (r, φ) the wave field is represented by the Sommerfeld integral

$$u(r, \varphi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{d}\alpha \; \mathrm{e}^{-\mathrm{i}kr\cos\alpha} f(\alpha + \varphi), \tag{16}$$

where γ is the Sommerfeld double-loop contour (Figure 4), $f(\,\cdot\,)$ is the so-called Sommerfeld transformant, which is a meromorphic function depending on the wave number k. The representation (16) is definitely valid as $|\varphi| \leqslant \Phi$ and also satisfies the Helmholtz equation in the whole exterior Ω of the polygon. In the same manner we introduce the representation of the solution in the polar coordinates attributed to A_3:

$$u(\rho_e, \varphi_e) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{d}\alpha \; \mathrm{e}^{-\mathrm{i}k\rho_e\cos\alpha} h(\alpha + \varphi_e), \tag{17}$$

[3] A solution of a Cauchy problem for an elliptic equation with the Cauchy data on a surface, located in the domain of ellipticity of the equation, is unique although such a problem is ill-posed in the Hadamard's sense.

where $h(\,\cdot\,)$ is the meromorphic Sommerfeld transformant of the solution of the Helmholtz equation in these coordinates, $-\Phi_e^- < \varphi_e < \pi - \Phi_2^-$. The representations (16) and (17) specify the same wave field in the overlapping domain of the angles $|\varphi| \leqslant \Phi$ and $-\Phi_e^- \leqslant \varphi_e \leqslant \pi - \Phi_2^-$. We also introduce the Sommerfeld integral representation of the wave field in the polar coordinates (ρ_2, φ_2) with the meromorphic Sommerfeld transformant $g(\,\cdot\,)$ for $-\Phi_2^- \leqslant \varphi_2 \leqslant \pi - \Phi$ that is attributed to A_2,

$$u(\rho_2, \varphi_2) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \mathrm{d}\alpha\, \mathrm{e}^{-\mathrm{i}k\rho_2 \cos\alpha} g(\alpha + \varphi_2).$$

Following the procedure described in Appendix,[4] see also [18], from the boundary conditions we arrive at a system of Malyuzhinets functional equations (written in three pairs) for the Sommerfeld transformants:

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_+) f(\alpha + \Phi) - (-\sin\alpha + \sin\vartheta_+) f(-\alpha + \Phi) = 0, \\
&(\sin\alpha - \sin\vartheta_1) f(\alpha - \Phi) - (-\sin\alpha - \sin\vartheta_1) f(-\alpha - \Phi) \\
&= \mathrm{e}^{\mathrm{i}kd_1 \cos\alpha} \big[(\sin\alpha - \sin\vartheta_1) g(\alpha - \Phi) - (-\sin\alpha - \sin\vartheta_1) g(-\alpha - \Phi)\big],
\end{aligned} \tag{18}$$

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_1) g(\alpha - [\Phi - \pi]) - (-\sin\alpha + \sin\vartheta_1) g(-\alpha - [\Phi - \pi]) \\
&= \mathrm{e}^{\mathrm{i}kd_1 \cos\alpha} \big[(\sin\alpha + \sin\vartheta_1) f(\alpha - [\Phi - \pi]) \\
&\qquad - (-\sin\alpha + \sin\vartheta_1) f(-\alpha - [\Phi - \pi])\big], \\
&(\sin\alpha - \sin\vartheta_2) g(\alpha - \Phi_2^-) - (-\sin\alpha - \sin\vartheta_2) g(-\alpha - \Phi_2^-) \\
&= \mathrm{e}^{\mathrm{i}kd_2 \cos\alpha} \big[(\sin\alpha - \sin\vartheta_2) h(\alpha - \Phi_2^-) \\
&\qquad - (-\sin\alpha - \sin\vartheta_2) h(-\alpha - \Phi_2^-)\big],
\end{aligned} \tag{19}$$

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_2) h(\alpha - [\Phi_2^- - \pi]) - (-\sin\alpha + \sin\vartheta_2) h(-\alpha - [\Phi_2^- - \pi]) \\
&= \mathrm{e}^{\mathrm{i}kd_2 \cos\alpha} \big[(\sin\alpha + \sin\vartheta_2) g(\alpha - [\Phi_2^- - \pi]) \\
&\qquad - (-\sin\alpha + \sin\vartheta_2) g(-\alpha - [\Phi_2^- - \pi])\big], \\
&(\sin\alpha - \sin\vartheta_-) h(\alpha - \Phi_e^-) - (-\sin\alpha - \sin\vartheta_-) h(-\alpha - \Phi_e^-) = 0.
\end{aligned} \tag{20}$$

Three connected pairs of the functional equations (18), (19), (20) should be supplemented by additional conditions specifying a class of meromorphic functions that also ensure Meixner's and radiation conditions for the total field represented by the Sommerfeld integrals. It can be verified that in the limiting case $d_{1,2} \to 0$, the system of equations (18), (19), (20) reduces to

[4] In Appendix we briefly discuss a way to derive the functional equations, and our approach methodologically differs from that in [18].

the well-known Malyuzhinets functional equations for one unknown function encountered in the problem of diffraction by an impedance wedge.

We assume that $f(\,\cdot\,)$ is regular in the strip

$$\Pi(-\Phi, \Phi) = \{\alpha \in \mathrm{C}\colon\; -\Phi < \Re(\alpha) < \Phi\},$$

i.e., it is holomorphic in this strip, having a simple pole at $\alpha = \varphi_s$ ($\varphi_s := \Phi - \vartheta_+$) on its boundary so that

$$f(\alpha) - \frac{1}{\alpha - \varphi_s}$$

is holomorphic in the strip $\Pi(-\Phi - \epsilon, \Phi + \epsilon)$ for any sufficiently small $\epsilon > 0$. Recall that such a condition is necessary in the framework of the Sommerfeld–Malyuzhinets technique and enables one to reproduce the incident wave in the far-field asymptotics when applying the steepest-descent method to the Sommerfeld integral and crossing the pole in the process of deformation of the contour γ into the steepest-descent paths. In order to ensure Meixner's condition at A_1, it is also assumed that $f(\mathrm{i}\infty) = -f(-\mathrm{i}\infty)$ are finite and $f(\alpha) - f(\pm\mathrm{i}\infty)$ are of $O(\exp(\pm\mathrm{i}\delta_1\alpha))$, $\delta_1 > 0$, i.e., they exponentially vanish as $\alpha \to \pm\mathrm{i}\infty$ and $\alpha \in \Pi(-\Phi, \Phi)$.

In a similar way, the transformant $h(\,\cdot\,)$ is meromorphic and $h(\alpha)$ is regular in the strip $\Pi(-\Phi_e^-, \pi - \Phi_2^-)$, $h(\mathrm{i}\infty) = -h(-\mathrm{i}\infty)$ is finite, and $h(\alpha) - h(\pm\mathrm{i}\infty)$ is of $O(\exp(\pm\mathrm{i}\alpha\delta_3))$ as $\alpha \to \pm\mathrm{i}\infty$ and $\alpha \in \Pi(-\Phi_e^-, \pi - \Phi_2^-)$. It is of interest to note that the meromorphic Sommerfeld transformants are interconnected by the relation

$$f(\alpha) = h(\alpha)\,\mathrm{e}^{\mathrm{i}k(d_1\cos[\alpha+\Phi]+d_2\cos[\alpha+\Phi_2^-])} + f_*(\alpha),$$

where $f_*(\alpha)$ is a function of $O(\exp(\mathrm{i}\delta_*\cos[\alpha+\delta]))$ in this strip, $\alpha \to \pm\mathrm{i}\infty$ for some constants δ_*, δ.

The transformant $g(\,\cdot\,)$ is meromorphic and $g(\alpha)$ is regular in the strip $\Pi(-\Phi_2^-, \pi - \Phi)$, $g(\mathrm{i}\infty) = -g(-\mathrm{i}\infty)$ is finite and $g(\alpha) - g(\pm\mathrm{i}\infty)$ is of $O(\exp(\pm\mathrm{i}\alpha\delta_2))$ as $\alpha \to \pm\mathrm{i}\infty$ and $\alpha \in \Pi(-\Phi_2^-, \pi - \Phi)$. The relation

$$g(\alpha) = h(\alpha)\,\mathrm{e}^{\mathrm{i}kd_2\cos[\alpha+\Phi_2^-]} + g_*(\alpha)$$

is valid, $g_*(\alpha)$ is a function of $O(\exp(\mathrm{i}\delta^*\cos[\alpha+\delta]))$, $\alpha \to \pm\mathrm{i}\infty$ for some constants δ^*, δ.

In this section we have performed an important step in the framework of the extended SM method, namely, we have reduced the boundary-value problem of diffraction in question to that for the system of Malyuzhinets functional equations on the complex plane in the special class of meromorphic functions described above. Having specified the Sommerfeld transformants, we recover the wave field in Ω by means of the Sommerfeld integral representations.

§4. Reduction to integral equations

Solution of a system of functional equations in quadratures is possible only in some very special cases because such a system for several unknowns is, in general, equivalent to a matrix Riemann–Hilbert problem. However, as an alternative, reduction of the problem to some well-studied problems like Fredholm integral equations is an efficient and, hence, well-accepted way.

4.1. Representation of the Sommerfeld transformants in terms of new unknowns. We first transform the system of equations (18), (19), (20) to that with a simpler left-hand side. To this end, we exploit auxiliary meromorphic solutions $\psi_f(\alpha), \psi_g(\alpha), \psi_h(\alpha)$ of the homogeneous functional equations (see the left-hand sides of (18), (19), (20))

$$\begin{aligned}(\sin\alpha + \sin\vartheta_+)\psi_f(\alpha+\Phi) - (-\sin\alpha + \sin\vartheta_+)\psi_f(-\alpha+\Phi) = 0,\\ (\sin\alpha - \sin\vartheta_1)\psi_f(\alpha-\Phi) - (-\sin\alpha - \sin\vartheta_1)\psi_f(-\alpha-\Phi) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}(\sin\alpha + \sin\vartheta_1)\psi_g(\alpha+\Phi_-) - (-\sin\alpha + \sin\vartheta_1)\psi_g(-\alpha+\Phi_-) = 0,\\ (\sin\alpha - \sin\vartheta_2)\psi_g(\alpha-\Phi_-) - (-\sin\alpha - \sin\vartheta_2)\psi_g(-\alpha-\Phi_-) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}(\sin\alpha + \sin\vartheta_2)\psi_h(\alpha+\Phi_+) - (-\sin\alpha + \sin\vartheta_2)\psi_h(-\alpha+\Phi_+) = 0,\\ (\sin\alpha - \sin\vartheta_-)\psi_h(\alpha-\Phi_+) - (-\sin\alpha - \sin\vartheta_-)\psi_h(-\alpha-\Phi_+) = 0,\end{aligned} \tag{23}$$

where

$$\Phi_+ = \frac{\pi - \Phi_2^- + \Phi_e^-}{2}, \quad \Phi_- = \frac{\pi - \Phi + \Phi_2^-}{2}.$$

The auxiliary meromorphic solutions $\psi_f(\alpha), \psi_g(\alpha), \psi_h(\alpha)$ of the respective equations (21), (22), (23) are represented as products of the Malyuzhinets function ([15, Chapter 6]) of appropriate arguments, where the latter is a specially normalized solution of the Malyuzhinets functional equation

$$\psi_\Phi(z+2\Phi)/\psi_\Phi(z-2\Phi) = \cot[z/2+\pi/4],$$

$$\begin{aligned}\psi_f(\alpha) = {}&\psi_\Phi(\alpha+\Phi+\pi/2-\vartheta_+)\psi_\Phi(\alpha+\Phi-\pi/2+\vartheta_+)\\ &\times \psi_\Phi(\alpha-\Phi+\pi/2-\vartheta_1)\psi_\Phi(\alpha-\Phi-\pi/2+\vartheta_1),\end{aligned}$$

$$\begin{aligned}\psi_g(\alpha) = {}&\psi_{\Phi_-}(\alpha+\Phi_-+\pi/2-\vartheta_1)\psi_{\Phi_-}(\alpha+\Phi_--\pi/2+\vartheta_1)\\ &\times \psi_{\Phi_-}(\alpha-\Phi_-+\pi/2-\vartheta_2)\psi_{\Phi_-}(\alpha-\Phi_--\pi/2+\vartheta_2),\end{aligned}$$

$$\begin{aligned}\psi_h(\alpha) = {}&\psi_{\Phi_+}(\alpha+\Phi_++\pi/2-\vartheta_2)\psi_{\Phi_+}(\alpha+\Phi_+-\pi/2+\vartheta_2)\\ &\times \psi_{\Phi_+}(\alpha-\Phi_++\pi/2-\vartheta_-)\psi_{\Phi_+}(\alpha-\Phi_+-\pi/2+\vartheta_-).\end{aligned}$$

These functions are respectively regular in the basic strips

$$\Pi(-\Phi,\Phi),\quad \Pi(-\Phi_-,\Phi_-),\quad \Pi(-\Phi_+,\Phi_+),$$

having neither poles nor zeros on the boundaries for passive impedances. They admit the estimates

$$\begin{aligned}&\psi_f(\alpha)=O(\cos\mu\alpha),\quad \psi_g(\alpha)=O(\cos\mu_-\alpha),\\&\psi_h(\alpha)=O(\cos\mu_+\alpha),\quad \Im(\alpha)\to\pm\infty\end{aligned}\tag{24}$$

in the respective strips with

$$\mu=\frac{\pi}{2\Phi},\quad \mu_-=\frac{\pi}{2\Phi_-},\quad \mu_+=\frac{\pi}{2\Phi_+}.$$

At the first stage of the reduction, we introduce the notation

$$\begin{aligned}&f_r(\alpha)=f(\alpha),\quad g_r(\alpha)=g\Big(\alpha+\frac{\pi-\Phi-\Phi_2^-}{2}\Big),\\&h_r(\alpha)=h\Big(\alpha+\frac{\pi-\Phi_e^--\Phi_2^-}{2}\Big),\end{aligned}$$

then write system (18), (19), (20) in the form

$$\begin{aligned}(\sin\alpha+\sin\vartheta_+)f_r(\alpha+\Phi)-(-\sin\alpha+\sin\vartheta_+)f_r(-\alpha+\Phi)&=\chi_f^+(\alpha),\\(\sin\alpha-\sin\vartheta_1)f_r(\alpha-\Phi)-(-\sin\alpha-\sin\vartheta_1)f_r(-\alpha-\Phi)&=\chi_f^-(\alpha),\end{aligned}$$

$$\begin{aligned}(\sin\alpha+\sin\vartheta_1)g_r(\alpha+\Phi_-)-(-\sin\alpha+\sin\vartheta_1)g_r(-\alpha+\Phi_-)&=\chi_g^+(\alpha),\\(\sin\alpha-\sin\vartheta_2)g_r(\alpha-\Phi_-)-(-\sin\alpha-\sin\vartheta_2)g_r(-\alpha-\Phi_-)&=\chi_g^-(\alpha),\end{aligned}$$

$$\begin{aligned}(\sin\alpha+\sin\vartheta_2)h_r(\alpha+\Phi_+)-(-\sin\alpha+\sin\vartheta_2)h_r(-\alpha+\Phi_+)&=\chi_h^+(\alpha),\\(\sin\alpha-\sin\vartheta_-)h_r(\alpha-\Phi_+)-(-\sin\alpha-\sin\vartheta_-)h_r(-\alpha-\Phi_+)&=\chi_h^-(\alpha),\end{aligned}$$

where the right-hand sides of the equations in (18), (19), (20) are denoted by $\chi_f^+(\alpha),\dots,\chi_h^-(\alpha)$. The last-mentioned are traditional Malyuzhinets equations for one unknown function and with known solutions (see [15, Chapter 6]), which is used for the reduction.

We can write the system (18), (19), (20) as follows

$$\begin{aligned}&\frac{f_r(\alpha+\Phi)}{\psi_f(\alpha+\Phi)}-\frac{f_r(-\alpha+\Phi)}{\psi_f(-\alpha+\Phi)}=0,\\&\frac{f_r(\alpha-\Phi)}{\psi_f(\alpha-\Phi)}-\frac{f_r(-\alpha-\Phi)}{\psi_f(-\alpha-\Phi)}=\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\alpha}}{\psi_f(\alpha-\Phi)}\,V_f(\alpha),\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{g_r(\alpha+\Phi_-)}{\psi_g(\alpha+\Phi_-)}-\frac{g_r(-\alpha+\Phi_-)}{\psi_g(-\alpha+\Phi_-)}&=\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\alpha}}{\psi_g(\alpha+\Phi_-)}V_g^+(\alpha),\\ \frac{g_r(\alpha-\Phi_-)}{\psi_g(\alpha-\Phi_-)}-\frac{g_r(-\alpha-\Phi_-)}{\psi_g(-\alpha-\Phi_-)}&=\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\alpha}}{\psi_g(\alpha-\Phi_-)}V_g^-(\alpha),\end{aligned}\tag{26}$$

$$\begin{aligned}\frac{h_r(\alpha+\Phi_+)}{\psi_h(\alpha+\Phi_+)}-\frac{h_r(-\alpha+\Phi_+)}{\psi_h(-\alpha+\Phi_+)}&=\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\alpha}}{\psi_h(\alpha+\Phi_+)}V_h(\alpha),\\ \frac{h_r(\alpha-\Phi_+)}{\psi_h(\alpha-\Phi_+)}-\frac{h_r(-\alpha-\Phi_+)}{\psi_h(-\alpha-\Phi_+)}&=0,\end{aligned}\tag{27}$$

where

$$V_f(\alpha)=g(\alpha-\Phi)+R_1^-(\alpha)g(-\alpha-\Phi),\tag{28}$$

$$V_g^+(\alpha)=f(\alpha+[\pi-\Phi])+R_1^+(\alpha)f(-\alpha+[\pi-\Phi]),\tag{29}$$

$$V_g^-(\alpha)=h(\alpha-\Phi_2^-)+R_2^-(\alpha)h(-\alpha-\Phi_2^-),\tag{30}$$

$$V_h(\alpha)=g(\alpha+[\pi-\Phi_2^-])+R_2^+(\alpha)g(-\alpha+[\pi-\Phi_2^-])\tag{31}$$

with

$$R_{1,2}^+(\alpha)=\frac{\sin\alpha-\sin\vartheta_{1,2}}{\sin\alpha+\sin\vartheta_{1,2}},\quad R_{1,2}^-(\alpha)=\frac{\sin\alpha+\sin\vartheta_{1,2}}{\sin\alpha-\sin\vartheta_{1,2}}$$

noticing that $R_j^+(\alpha)=[R_j^-(\alpha)]^{-1}$, $j=1,2$. $V_f(\alpha),V_g^{\pm}(\alpha),V_h(\alpha)$ exponentially vanish on the imaginary axis as $\alpha\to\pm\mathrm{i}\infty$.

Remark 4.1. For technical reasons, we assume that small $\Re(\vartheta_2)$ is strictly positive and view the case of $\Re(\vartheta_2)=0$ as a limiting case. In this case, $V_f,V_g^{\pm},V_h$ are regular in some neighborhood of the imaginary axis as well as the right-hand sides in (25), (26), (27), which also exponentially vanish as $\alpha\to\pm\mathrm{i}\infty$.

It can easily be verified that

$$\begin{aligned}V_f(-\alpha)&=V_f(\alpha)/R_1^-(\alpha),\quad V_g^+(-\alpha)=V_g^+(\alpha)/R_1^+(\alpha),\\ V_g^-(-\alpha)&=V_g^-(\alpha)/R_2^-(\alpha),\quad V_h(-\alpha)=V_h(\alpha)/R_2^+(\alpha).\end{aligned}\tag{32}$$

The difference operators on the left-hand side of equations (25), (26), (27) can be "inverted", i.e., the equations can be solved assuming that the right-hand sides are known. This can be done by means of the so-called S-integrals, a well-known technique discussed in Chapter 7 of [15]. This is actually equivalent to the Fourier transform along the imaginary axis. In this way, we have[5]

$$
\begin{aligned}
f_r(\alpha) &= -\frac{\mathrm{i}\psi_f(\alpha)}{8\Phi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau\,\sin\mu\tau}{\cos\mu\tau+\sin\mu\alpha}\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}}{\psi_f(\tau-\Phi)}V_f(\tau)+f_r^i(\alpha), \quad \alpha\in\Pi(-\Phi,\Phi),\\
g_r(\alpha) &= \frac{\mathrm{i}\psi_g(\alpha)}{8\Phi_-}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}\tau\left(\frac{V_g^+(\tau)\sin\mu_-\tau}{\cos\mu_-\tau-\sin\mu_-\alpha}\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}}{\psi_g(\tau+\Phi_-)}\right.\\
&\qquad\left.-\frac{V_g^-(\tau)\sin\mu_-\tau}{\cos\mu_-\tau+\sin\mu_-\alpha}\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}}{\psi_g(\tau-\Phi_-)}\right), \quad \alpha\in\Pi(-\Phi_-,\Phi_-),\\
h_r(\alpha) &= \frac{\mathrm{i}\psi_h(\alpha)}{8\Phi_+}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau\,\sin\mu_+\tau}{\cos\mu_+\tau-\sin\mu_+\alpha}\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}}{\psi_h(\tau+\Phi_+)}V_h(\tau), \quad \alpha\in\Pi(-\Phi_+,\Phi_+),
\end{aligned}
\tag{33}
$$

where the integral terms are holomorphic in the respective basic strips,

$$
f_r^i(\alpha) = \frac{\psi_f(\alpha)}{\psi_f(\varphi_s)}\frac{\mu\cos\mu\varphi_s}{\sin\mu\alpha-\sin\mu\varphi_s} = f_i(\alpha)
$$

has the pole on the boundary of the strip $\Pi(-\Phi,\Phi)$, $\varphi_s=\Phi-\vartheta_+$ with unit residue. The term f_r^i in (33) is due to the incident surface wave, it solves the respective homogeneous equations for f_r, however.

In the representations (33) we return to the Sommerfeld transformants f, g, h, thus having

$$
\begin{aligned}
f(\alpha) &= -\frac{\mathrm{i}\psi_f(\alpha)}{8\Phi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau\sin\mu\tau}{\cos\mu\tau+\sin\mu\alpha}\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}}{\psi_f(\tau-\Phi)}V_f(\tau)+f_i(\alpha), \quad \alpha\in\Pi(-\Phi,\Phi),\\
g(\alpha) &= \frac{\mathrm{i}\psi_g\left(\alpha-\frac{\pi-\Phi-\Phi_2^-}{2}\right)}{8\Phi_-}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}\tau\left(\frac{V_g^+(\tau)\sin\mu_-\tau}{\cos\mu_-\tau-\sin\mu_-\left(\alpha-\frac{\pi-\Phi-\Phi_2^-}{2}\right)}\frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}}{\psi_g(\tau+\Phi_-)}\right.\\
&\qquad\left.-\frac{V_g^-(\tau)\sin\mu_-\tau}{\cos\mu_-\tau+\sin\mu_-\left(\alpha-\frac{\pi-\Phi-\Phi_2^-}{2}\right)}\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}}{\psi_g(\tau-\Phi_-)}\right), \quad \alpha\in\Pi(-\Phi_2^-,\pi-\Phi),\\
h(\alpha) &= \frac{\mathrm{i}\psi_h\left(\alpha-\frac{\pi-\Phi_e^--\Phi_2^-}{2}\right)}{8\Phi_+}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau\sin\mu_+\tau}{\cos\mu_+\tau-\sin\mu_+\left(\alpha-\frac{\pi-\Phi_e^--\Phi_2^-}{2}\right)}\frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}}{\psi_h(\tau+\Phi_+)}V_h(\tau),\\
&\qquad \alpha\in\Pi(-\Phi_e^-,\pi-\Phi_2^-),
\end{aligned}
\tag{34}
$$

[5]Due to the parity of the integrands, the integrations below can be reduced to $\frac{\mathrm{i}}{4\Phi}\int_0^{\mathrm{i}\infty}(\dots)$.

where the last integral representations enable one to compute the Sommerfeld transformants in the respective basic strips on the complex plane provided that the functions $V_f(\,\cdot\,), V_g^{\pm}(\,\cdot\,), V_h(\,\cdot\,)$ are known on the imaginary axis. This observation shows that it is desirable to get integral equations specifying $V_f(\,\cdot\,), V_g^{\pm}(\,\cdot\,), V_h(\,\cdot\,)$ in a neighborhood of the imaginary axis.

Remark that the first representation in (34) is actually valid as

$$\alpha \in \Pi(-\Phi, 3\Phi),$$

whereas the last one is as

$$\alpha \in \Pi(-\Phi_e^- - 2\Phi_+, \pi - \Phi_2^-).$$

In this respect it is useful to notice that the right-hand side in the expression for f has poles at $\alpha = \varphi_s$ and $\alpha = \pi + \Phi + \vartheta_+$, where the latter is the pole of $\psi_f(\alpha)$. It is responsible for the surface wave outgoing at infinity along l^1 and reflected from A_1. Similarly, the pole of $\psi_h\big(\alpha - \frac{\pi - \Phi_e^- - \Phi_2^-}{2}\big)$ is at $\alpha = -(\pi + \Phi_e^- + \vartheta_-)$. It specifies the outgoing surface wave along l^4. As a result, the radiation condition is expected to be satisfied.

4.2. Integral equations for $V_f, V_g^{\pm}, V_h$. We substitute the representations for the Sommerfeld transformants in (34) in equalities (28)–(31). It is important that the arguments of the functions f, g, h on the right-hand sides of (28)–(31) belong to the basic strips on the complex plane, where the respective representations (34) are valid.[6]

$$\begin{aligned}
V_f(\alpha) &= V_f^i(\alpha) \\
&+ \frac{\mathrm{i}}{8\Phi_-}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \Bigg\{ \Bigg(\frac{\psi_g\left(\alpha - \frac{\Phi+\pi-\Phi_2^-}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)} \\
&+ R_1^-(\alpha)\frac{\psi_g\left(-\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(-\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)} \Bigg) V_g^+(\tau) \\
&- \Bigg(\frac{\psi_g\left(\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)}{\psi_g(\tau-\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau + \sin\mu_-\left(\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)} \\
&+ R_1^-(\alpha)\frac{\psi_g\left(-\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)}{\psi_g(\tau-\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau + \sin\mu_-\left(-\alpha + \frac{\Phi_2^- - \pi - \Phi}{2}\right)} \Bigg) V_g^-(\tau) \Bigg\},
\end{aligned} \tag{35}$$

[6] Exactly at this place we are compelled to make use of the limitations on the geometry of the scatterer, see (6), otherwise, our hint of the derivation is not directly applicable, because some of the arguments are not in the basic strips.

$$
\begin{aligned}
V_g^+(\alpha) = {} & V_g^{+,i}(\alpha) \\
& + \frac{-\mathrm{i}}{8\Phi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \Bigg\{ \Bigg(\frac{\psi_f\,(\alpha-\Phi+\pi)}{\psi_f(\tau-\Phi)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu\tau}{\cos\mu\tau+\sin\mu\,(\alpha-\Phi+\pi)} \\
& + R_1^+(\alpha)\frac{\psi_f\,(-\alpha-\Phi+\pi)}{\psi_f(\tau-\Phi)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu\tau}{\cos\mu\tau+\sin\mu\,(-\alpha-\Phi+\pi)} \Bigg) V_f(\tau) \Bigg\},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
& V_g^-(\alpha) = V_g^{-,i}(\alpha) \\
& + \frac{\mathrm{i}}{8\Phi_+}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \Bigg\{ \Bigg(\frac{\psi_h\left(\alpha+\frac{\Phi_e^- -\pi-\Phi_2^-}{2}\right)}{\psi_h(\tau+\Phi_+)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_+\tau}{\cos\mu_+\tau-\sin\mu_+\left(\alpha+\frac{\Phi_e^- -\pi-\Phi_2^-}{2}\right)} \\
& + R_2^-(\alpha)\frac{\psi_h\left(-\alpha+\frac{\Phi_e^- -\pi-\Phi_2^-}{2}\right)}{\psi_h(\tau+\Phi_+)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_+\tau}{\cos\mu_+\tau-\sin\mu_+\left(-\alpha+\frac{\Phi_e^- -\pi-\Phi_2^-}{2}\right)} \Bigg) V_h(\tau) \Bigg\},
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
& V_h(\alpha) = V_h^i(\alpha) \\
& + \frac{\mathrm{i}}{8\Phi_-}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \Bigg\{ \Bigg(\frac{\psi_g\left(\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau-\sin\mu_-\left(\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \\
& + R_2^+(\alpha)\frac{\psi_g\left(-\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau-\sin\mu_-\left(-\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \Bigg) V_g^+(\tau) \\
& - \Bigg(\frac{\psi_g\left(\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau-\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau+\sin\mu_-\left(\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \\
& + R_2^+(\alpha)\frac{\psi_g\left(-\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau-\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_2\cos\tau}\sin\mu_-\tau}{\cos\mu_+\tau+\sin\mu_-\left(-\alpha+\frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \Bigg) V_g^-(\tau) \Bigg\},
\end{aligned} \tag{38}
$$

where the free terms are

$$
\begin{aligned}
V_f^i(\alpha) = 0, \quad & V_g^{+,i}(\alpha) = f_i(\alpha+[\pi-\Phi]) + R_1^+(\alpha) f_i(-\alpha+[\pi-\Phi]), \\
V_g^{-,i}(\alpha) = 0, \quad & V_h^i(\alpha) = 0.
\end{aligned}
$$

It should be noted that the representations (35), (36), (37), (38) are valid as $\alpha \in \Pi(-\epsilon,\epsilon)$, $\epsilon > 0$ is small, i.e., α belongs to some narrow strip containing the imaginary axis iR. Now, we let $\alpha \to \mathrm{iR}$ and take into account the evenness of the integrands, which is true in view of (21)–(23) and (32). We arrive at the integral equations

$$
\begin{aligned}
V_f(\alpha) &= V_f^i(\alpha) + \int_0^{i\infty} d\tau \left(\mathcal{K}_{12}(\alpha,\tau;k)V_g^+(\tau) + \mathcal{K}_{13}(\alpha,\tau;k)V_g^-(\tau)\right),\\
V_g^+(\alpha) &= V_g^{+,i}(\alpha) + \int_0^{i\infty} d\tau\, \mathcal{K}_{21}(\alpha,\tau;k)V_f(\tau),\\
V_g^-(\alpha) &= V_g^{-,i}(\alpha) + \int_0^{i\infty} d\tau\, \mathcal{K}_{34}(\alpha,\tau;k)V_h(\tau),\\
V_h(\alpha) &= V_h^i(\alpha) + \int_0^{i\infty} d\tau \left(\mathcal{K}_{42}(\alpha,\tau;k)V_g^+(\tau) + \mathcal{K}_{43}(\alpha,\tau;k)V_g^-(\tau)\right),
\end{aligned}
\tag{39}
$$

where the expressions of the kernel entries $\mathcal{K}_{ij}(\alpha,\tau;k)$ in (39) are obvious from (35)–(38) (not forgetting to multiply the integrands by 2). Recall that free terms in the equations are zero except $V_g^{+,i}(\alpha)$. Having obtained solutions of (39) on $i\mathrm{R}_+$, we extend them to the whole imaginary axis by means of (32).

We notice that, provided $\alpha \in \Pi(-\epsilon,\epsilon)$, the representations (39) enables one to compute V as holomorphic functions (see Remark 4.1) in this strip having their values on the imaginary axis on the right-hand side.

The system of integral equations is written in the matrix form

$$
\begin{pmatrix} V_f \\ V_g^+ \\ V_g^- \\ V_h \end{pmatrix} = \begin{pmatrix} 0 & K_{12} & K_{13} & 0 \\ K_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{34} \\ 0 & K_{42} & K_{43} & 0 \end{pmatrix} \begin{pmatrix} V_f \\ V_g^+ \\ V_g^- \\ V_h \end{pmatrix} + \begin{pmatrix} V_f^i \\ V_g^{+,i} \\ V_g^{-,i} \\ V_h^i \end{pmatrix}
$$

or

$$
V = KV + V_i, \tag{40}
$$

where $K=\{K_{ij}\}_{i,j=1}^4$ is the matrix integral operator with the entries $\mathcal{K}_{ij}(\alpha,\tau;k)$ of the kernel,

$$
V = (V_f(\alpha), V_g^+(\alpha), V_g^-(\alpha), V_h(\alpha))^t, \quad V_i = (V_f^i(\alpha), V_g^{+,i}(\alpha), V_g^{-,i}(\alpha), V_h^i(\alpha))^t.
$$

The integral equation (40) is of the second kind and its kernel depends on the wave number k, which plays the role of the *characteristic* parameter. It is easily verified that $V_i(\,\cdot\,)$ is a holomorphic vector if $\alpha \in \Pi(-\epsilon,\epsilon)$ and admits meromorphic continuation on the complex plane. Moreover, its components exponentially vanish as $\alpha \to \pm i\infty$ in $\Pi(-\epsilon,\epsilon)$, which follows from the explicit formulas for the components having the properties described in §3, and

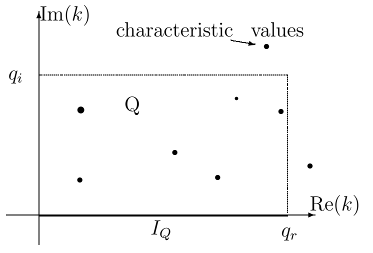

Figure 3. Domain of the characteristic parameter k.

from (24). As a result, the components of V_i are from $L_2(\mathrm{iR})$ and also from $L_2(\mathrm{iR}_+)$,

$$V_i \in \mathcal{L}_2(\mathrm{iR}_+) := L_2(\mathrm{iR}_+) \oplus L_2(\mathrm{iR}_+) \oplus L_2(\mathrm{iR}_+) \oplus L_2(\mathrm{iR}_+).$$

From the expressions (28)–(31) and asymptotics of their right-hand sides, we find

$$V \in \mathcal{L}_2(\mathrm{iR}_+).$$

In the next section we study the integral equation (40) in $\mathcal{L}_2(\mathrm{iR}_+)$, taking into account that it depends on the characteristic parameter k. It is crucial to prove the Fredholm property of equation (40) and study the properties of the integral operator

$$K\colon \mathcal{L}_2(\mathrm{iR}_+) \to \mathcal{L}_2(\mathrm{iR}_+),$$

which enables one to study its solvability.

Observe that the Fredholm property of the integral equation (40) is actually ensured by the presence of factors $\mathrm{e}^{\mathrm{i}kd_{1,2}\cos\tau}$ in the kernel which rapidly vanish as $\tau \to \mathrm{i}\infty$ and $\Im(k) > 0$. However, it is also possible to study the case of $\Im(k) = 0$.

§5. Fredholm property and solvability of the integral equation

We intend to apply the analytic Fredholm alternative (see, e.g., [23, Theorems 2, 3 of Subsection 2.8]) to the integral equation (40). We introduce the rectangular domain Q on the complex plane of the characteristic parameter k (wave number)

$$Q = \{k \in \mathrm{C}\colon 0 < \Re(k) < q_r, \quad 0 < \Im(k) < q_i\}$$

and denote by $I_Q = \{k \in \mathrm{C}\colon 0 < \Re(k) < q_r, \Im(k) = 0\}$ the open interval of the real axis that is the lower side of the rectangle Q, Figure 3. The integral operator K in equation (40) depends on the characteristic parameter k. We assume, in this section, that $k \in Q$, i.e., in particular, $\Im(k) > 0$ or $k \in I_Q$ (i.e., $\Im(k) = 0$) as a limit from the rectangle Q.

Assume that $k \in Q$ and consider the estimate, for instance, for the operator K_{42} that is an entry of the matrix operator K, observing that the other entries are evaluated in a similar manner,

$$|K_{42}V(\alpha)| = \left| \frac{\mathrm{i}}{4\Phi_-} \int_0^{\mathrm{i}\infty} \mathrm{d}\tau \left(\frac{\psi_g\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right.\right.$$
$$\left.\left. + R_2^+(\alpha) \frac{\psi_g\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right) V(\tau) \right|$$
$$\leqslant J(\alpha)\|V\|_{L_2(\mathrm{iR})},$$

where

$$J^2(\alpha) = \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau|\,|\mathcal{K}_{42}(\alpha,\tau;k)|^2$$
$$\leqslant \int_0^{\mathrm{i}\infty} \frac{|\mathrm{d}\tau|}{(4\Phi_-)^2} \left| \left(\frac{\psi_g(\alpha - \Phi_- + \chi)}{\psi_g(\tau + \Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau + \cos\mu_-(\alpha+\chi)} \right.\right.$$
$$\left.\left. + R_2^+(\alpha) \frac{\psi_g(-\alpha - \Phi_- + \chi)}{\psi_g(\tau + \Phi_-)} \frac{\mathrm{e}^{\mathrm{i}kd_1\cos\tau}\sin\mu_-\tau}{\cos\mu_-\tau + \cos\mu_-(-\alpha+\chi)} \right) \right|^2$$

with $\chi = \pi$. $J(\alpha)$ is a continuous function of $\alpha \in \mathrm{iR}$, and we consider its behavior at infinity $\alpha \to \pm\mathrm{i}\infty$. To this end, we make use of

$$\psi_g(-\alpha - \Phi_- + \chi) = -\mathrm{e}^{\pm\mathrm{i}2\chi\mu_-}\psi_g(\alpha - \Phi_- + \chi)\left(1 + O(\mathrm{e}^{-\delta_0|\Im(\alpha)|})\right), \quad \delta_0 > 0,$$

thus having

$$J^2(\alpha) \leqslant C_1 \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \mathrm{e}^{-2\Im(kd_1)\cosh|\tau|} \left| \frac{\psi_g(\alpha - \Phi_- + \chi)}{\psi_g(\tau + \Phi_-)} \right|^2$$
$$\times \left| \left(\frac{\sin\mu_-\tau}{\cos\mu_-\tau + \cos\mu_-(\alpha+\chi)} - R_2^+(\alpha) \frac{\mathrm{e}^{\pm\mathrm{i}2\chi\mu_-}\sin\mu_-\tau}{\cos\mu_-\tau + \cos\mu_-(\alpha-\chi)} \right) \right|^2$$
$$\leqslant C_2 |\psi_g(\alpha - \Phi_- + \chi)|^2 \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \left| \frac{\sin\mu_-\tau}{\cos\mu_-\tau} \right|^2 \mathrm{e}^{-2\Im(kd_1)\cosh|\tau|}$$

$$\times\left|\left(\frac{\mathrm{e}^{\mp\mathrm{i}\chi\mu_-}}{\cos\mu_-\tau+\cos\mu_-(\alpha+\chi)}-\frac{\mathrm{e}^{\pm\mathrm{i}\chi\mu_-}}{\cos\mu_-\tau+\cos\mu_-(\alpha-\chi)}\right)\right|^2$$

$$\leqslant C_3\mathrm{e}^{-\Im(kd_1)}\int_0^{\mathrm{i}\infty}|\mathrm{d}\tau|\mathrm{e}^{-\Im(kd_1)\cosh|\tau|}$$

$$\times\left|\left(\frac{\mathrm{e}^{\mp\mathrm{i}\chi\mu_-}|\psi_g(\alpha-\Phi_-+\chi)|}{\cos\mu_-\tau+\cos\mu_-(\alpha+\chi)}-\frac{\mathrm{e}^{\pm\mathrm{i}\chi\mu_-}|\psi_g(\alpha-\Phi_-+\chi)|}{\cos\mu_-\tau+\cos\mu_-(\alpha-\chi)}\right)\right|^2,$$

where we have exploited the asymptotics (24). After simple transformations, we find

$$J^2(\alpha)\leqslant C_4\mathrm{e}^{-\Im(kd_1)}\int_0^{\mathrm{i}\infty}|\mathrm{d}\tau|\mathrm{e}^{-\Im(kd_1)\cosh|\tau|}|\cos\mu_-\tau|^2$$

$$\times\left|\frac{\psi_g(\alpha-\Phi_-+\chi)}{[\cos\mu_-\tau+\cos\mu_-(\alpha+\chi)][\cos\mu_-\tau+\cos\mu_-(\alpha-\chi)]}\right|^2$$

and

$$J^2(\alpha)\leqslant C_5\frac{\mathrm{e}^{-\Im(kd_1)}}{|\cos\mu_-\alpha|^2}$$

$$\times\int_0^{\mathrm{i}\infty}|\mathrm{d}\tau|\mathrm{e}^{-\Im(kd_1)\cosh|\tau|}|\cos\mu_-\tau|^2\left|\frac{\psi_g(\alpha-\Phi_-+\chi)}{[\cos\mu_-\tau+\cos\mu_-(\alpha+\chi)]}\right|^2,$$

where the integral on the right-hand side is bounded uniformly with respect to $\alpha\in\mathrm{iR}$. As a result,

$$J^2(\alpha)\leqslant C\frac{\mathrm{e}^{-\Im(kd_1)}}{|\cos\mu_-\alpha|^2},$$

and

$$\|K_{42}V\|_{L_2(\mathrm{iR}_+)}\leqslant C_J\mathrm{e}^{-\frac{1}{2}\Im(kd_1)}\|V\|_{L_2(\mathrm{iR}_+)}.$$

Quite similar estimates are valid for the other entries of the operator K.

Lemma 5.1. *Suppose that $k\in Q$ and $d=\min\{d_1,d_2\}$, then the operator K is of the Hilbert–Schmidt class (i.e., compact) and*

$$\|K\|_{\mathcal{L}_2(\mathrm{iR}_+)\to\mathcal{L}_2(\mathrm{iR}_+)}\leqslant C_*(k)\mathrm{e}^{-\frac{1}{2}\Im(kd)},$$

where the constant C_* depends on the characteristic parameter k and is bounded for any compact domain Q.

Remark that the estimate fails as $\Im(k)\to 0$. Taking $k\in Q$ with $\Im(k)$ sufficiently large, provided that q_i is large, such that from Lemma 5.1, one has $\|K\|_{\mathcal{L}_2(\mathrm{iR}_+)\to\mathcal{L}_2(\mathrm{iR}_+)}<1$, and we arrive at the following claim.

Lemma 5.2. *There exists $k \in Q$ such that the operator K is a contraction and then $(I - K(k))^{-1}$ is bounded in $\mathcal{L}_2(\mathrm{iR}_+)$.*

Now, we assume that $k \in Q \cup I_Q$ (i.e., k may be real positive) and analogously obtain the estimate

$$|K_{42}V(\alpha)| \leqslant J_0(\alpha)\|V\|_{L_2\mathrm{iR}_+},$$

where

$$J_0^2(\alpha) \leqslant C_1 \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \left|\frac{\sin\mu_-\tau}{\cos\mu_-\tau}\right|^2 \times \left|\left(\frac{\mathrm{e}^{\mp\mathrm{i}\chi\mu_-}\cos\mu_-\alpha}{\cos\mu_-\tau + \cos\mu_-(\alpha+\chi)} - \frac{\mathrm{e}^{\pm\mathrm{i}\chi\mu_-}\cos\mu_-\alpha}{\cos\mu_-\tau + \cos\mu_-(\alpha-\chi)}\right)\right|^2$$

$$\leqslant C_2 \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \left|\frac{\sin\mu_-\tau}{\cos\mu_-\tau}\right|^2 \times \left|\left(\frac{\mathrm{e}^{\mp\mathrm{i}\chi\mu_-}}{\cos\mu_-\tau/\cos\mu_-\alpha + \mathrm{e}^{\mp\mathrm{i}\chi\mu_-}} - \frac{\mathrm{e}^{\pm\mathrm{i}\chi\mu_-}}{\cos\mu_-\tau/\cos\mu_-\alpha + \mathrm{e}^{\pm\mathrm{i}\chi\mu_-}}\right)\right|^2.$$

Changing the variable of integration in the last integral to $t = \frac{\cos\mu_-\tau}{\cos\mu_-\alpha}$, we have

$$J_0^2(\alpha) \leqslant C_3 \int_{\frac{1}{\cos\mu_-\alpha}}^{\mathrm{i}\infty} \frac{\mathrm{d}t}{t} \left|\left(\frac{1}{1+t\mathrm{e}^{\pm\mathrm{i}\chi\mu_-}} - \frac{1}{1+t\mathrm{e}^{\mp\mathrm{i}\chi\mu_-}}\right)\right|^2 \leqslant C^4$$

if $\alpha \in \mathrm{iR}_+$. Next $[K_{42}V](\,\cdot\,)$ is a continuous function for $\alpha \in \mathrm{iR}_+$. Exploiting analogous estimates for the other entries of K, we arrive at another lemma.

Lemma 5.3. *The operator $K\colon \mathcal{L}_2(\mathrm{iR}_+) \to C(\mathrm{iR}_+)$ is bounded as $k \in Q \cup I_Q$ and*

$$\|KV\|_{C(\mathrm{iR}_+)} \leqslant C^*\|V\|_{\mathcal{L}_2(\mathrm{iR}_+)},$$

where C^* can be taken independent of k.

It should also be observed that the kernel of the operator K depends on the characteristic parameter k analytically if $k \in Q$ and one can verify that

$$\lim_{\triangle k\to 0} \int_0^{\mathrm{i}\infty} |\mathrm{d}\alpha| \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \left|\frac{\mathcal{K}_{ij}(\alpha,\tau;k+\triangle k) - \mathcal{K}_{ij}(\alpha,\tau;k)}{\triangle k} - \frac{\partial}{\partial k}\mathcal{K}_{ij}(\alpha,\tau;k)\right|^2 = 0.$$

The last identity follows from the respective estimate, which is proved much as that for $J(\alpha)$ above. Also, taking into account that for any $k \in Q$ we have

$$\|K_{ij}\|_{L_2(\mathrm{iR}_+)\to L_2(\mathrm{iR})} \leqslant \|\mathcal{K}_{ij}\|_{L_2(\mathrm{iR}_+)}, \quad i,j = 1,\ldots,4,$$

we finally conclude the following.

Lemma 5.4. *The operator-function $K(k)$ is holomorphic for $k \in Q$.*

Now we are ready to apply the analytic Fredholm alternative to the operator K (see [23, Theorem 2, Subsection 2.8]). We introduce the set $N = \{k \in Q : 1 \in \sigma(K(k))\}$.[7] Such values of k are called characteristic. Thus we have the following statement.

Theorem 5.1. *Let $k \in Q$, then the operator $K(k)$ in (40) is Fredholm, i.e., $(I - K(k))^{-1}$ is meromorphic in Q and, if $k \in Q \setminus N$, then $(I - K(k))^{-1}$ is bounded.*

For noncharacteristic values of $k \in Q$ the integral equation (40) is uniquely solvable. For the scattering theory, however, real values of the characteristic parameter are of principal importance. In order to study the Fredholm property as $k \in I_Q$, we take into account that V_i vanishes exponentially as $\alpha \to \mathrm{i}\infty$. It is then natural to look for solution of (40) in a class of exponentially vanishing functions (a weighted L_2 space),

$$\|V\|^2_{\mathcal{L}_{2,w}(\mathrm{iR}_+)} = \int_0^{\mathrm{i}\infty} |\mathrm{d}\alpha| w(\alpha) \|V(\alpha)\|^2_{\mathrm{C}^4} < \infty,$$

where $w(\alpha) = |\cos(\mu_* \alpha)|$ is the weight $(0 < \mu_* < \min\{\mu, \mu_+, \mu_-\})$.

Now, it is convenient to consider the operator

$$K_w \colon \mathcal{L}_{2,w}(\mathrm{iR}_+) \to \mathcal{L}_2(\mathrm{iR}_+).$$

It is obvious that $K_w = K|_{\mathcal{L}_{2,w}(\mathrm{iR}_+)}$ and $\mathcal{L}_{2,w}(\mathrm{iR}_+)$ is a subspace in $\mathcal{L}_2(\mathrm{iR}_+)$. Then we can verify the following claim.

Lemma 5.5. *For any $k \in Q \cup I_Q$, the estimate*

$$\|K_w(k)V\| \leqslant C_w \|V\|_{\mathcal{L}_{2,w}(\mathrm{iR}_+)}$$

is valid uniformly with respect to k.

Following the derivations above, we can show that $(I - K_w(k))^{-1}$ is meromorphic for $k \in Q$. We then also prove that $K_w(k)$ is continuous in the operator norm up to the segment I_Q. We make use of the next fact.

Lemma 5.6. *For any $k \in I_Q$ and $\epsilon \in [0, \epsilon_0)$ the estimate*

$$\|[K_w(k + \mathrm{i}\epsilon) - K_w(k)]V\| \leqslant J_w(k, \epsilon) \|V\|_{\mathcal{L}_{2,w}(\mathrm{iR}_+)}$$

[7] $\sigma(A)$ is the spectrum of a closed operator A, see Subsection 2.7 in [23].

is valid, where

$$J_w(k,\epsilon) \leqslant C_* \int_0^{\mathrm{i}\infty} |\mathrm{d}\alpha| \int_0^{\mathrm{i}\infty} |\mathrm{d}\tau| \frac{|\cos\mu_-\tau|^2}{w(\tau)}$$
$$\times \left| \frac{E(\tau,k,\epsilon)\cos\mu_-\alpha}{[\cos\mu_-\tau + \cos\mu_-(\alpha+\chi)][\cos\mu_-\tau + \cos\mu_-(\alpha-\chi)]} \right|^2$$

and $E(\tau,k,\epsilon) = \mathrm{e}^{\mathrm{i}[k+\mathrm{i}\epsilon]d_1\cos\tau} - \mathrm{e}^{\mathrm{i}kd_1\cos\tau}$.

The double integral on the right-hand side of the last estimate converges uniformly with respect to k, ϵ, which is followed by $J_w(k,\epsilon) \to 0$ as $\epsilon \to 0+$, $k \in Q \cup I_Q$. The lemma is verified by means of the estimates

$$\|[K_{42}(k+\mathrm{i}\epsilon) - K_{42}(k)]V(\alpha)\|^2$$
$$= \int_0^{\mathrm{i}\infty} |\mathrm{d}\alpha| \left| \frac{\mathrm{i}}{4\Phi_-} \int_0^{\mathrm{i}\infty} \frac{\mathrm{d}\tau}{\sqrt{w(\tau)}} \left(\frac{\psi_g\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{E(\tau,k,\epsilon)\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right.\right.$$
$$\left.\left. + R_2^+(\alpha) \frac{\psi_g\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{E(\tau,k,\epsilon)\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right) \sqrt{w(\tau)}V(\tau) \right|^2$$
$$\leqslant J_w^2(k,\epsilon)\|V\|^2_{L_{2,w}(\mathrm{iR})},$$

where we have used the Cauchy inequality and the notation

$$J_w^2(k,\epsilon)$$
$$= \int_0^{\mathrm{i}\infty} \frac{|\mathrm{d}\alpha|}{16\Phi_-^2} \int_0^{\mathrm{i}\infty} \frac{|\mathrm{d}\tau|}{w(\tau)} \left| \left(\frac{\psi_g\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{E(\tau,k,\epsilon)\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right.\right.$$
$$\left.\left. + R_2^+(\alpha) \frac{\psi_g\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)}{\psi_g(\tau+\Phi_-)} \frac{E(\tau,k,\epsilon)\sin\mu_-\tau}{\cos\mu_-\tau - \sin\mu_-\left(-\alpha + \frac{\Phi+[\pi-\Phi_2^-]}{2}\right)} \right) \right|^2.$$

The estimate for $J_w^2(k,\epsilon)$ is obtained much as in the proof of Lemma 5.1. The estimates for the other entries of K_w are derived in a similar manner.

We introduce also the set $N_+ = \{k \in \bar{I}_Q : 1 \in \sigma(K(k+\mathrm{i}0))\}$. Now we apply Lemma 5.6 and Theorem 3 in Subsection 2.8 of [23] and arrive at the following fact.

Theorem 5.2. *Let* $k \in (Q \cup I_Q) \setminus (N \cup N_+)$, *then* $(I - K_w(k))^{-1}$ *is bounded and equation* (40) *is uniquely solvable.*

Theorem 5.2 means that we have the set $N \cup N_+$ of zero Lebesgue measure of exceptional (characteristic) values of the parameter k such that the integral equation has a unique solution for all $k \in Q \cup I_Q$ except for those values.

Remark 5.1. One might expect that the set N_+ is empty because for positive k the uniqueness theorem for a solution of the scattering problem is valid. Indeed, any nontrivial solution to the homogeneous integral equation gives rise to a nontrivial solution of the homogeneous boundary-value problem (i.e., with $U^i = 0$), which contradicts the uniqueness of a classical solution. Some details of the proof of the unique solvability of the scattering problem in question require additional work, which will be presented elsewhere. It is also of interest to mention that solutions given in §7 were obtained for all positive k so that the numerical procedure used in this work, see §7, encounters no difficulties.

§6. The far-field asymptotics of the solution

Having obtained the solution $V \in \mathcal{L}_2(\mathrm{iR})$, we then continue it to the strip $\Pi(-\varepsilon, \varepsilon)$, i.e., to some neighborhood of the imaginary axis as was discussed above. On the other hand, formulas (34) enable us to compute the Sommerfeld transformants f, g, h in the respective basic strips, where the integral terms in their expressions are holomorphic. Recall that meromorphic continuation of the Sommerfeld transformants (see below) is performed by means of the functional equations (18)–(20) that we shall use in what follows. It is of value to note, however, that an alternative way of meromorphic continuation may be based on the respective continuation of the integrals in (34) by making use of properties of the S-integrals (see [15, Chapter 7]). We arrive at our next lemma.

Lemma 6.1. *Any solution of the integral equation* (40) *gives rise to the respective solution* f, g, h *of the Malyuzhinets functional equations* (18)–(20) *in the prescribed class of meromorphic functions.*

Consider the Sommerfeld representation (16) of the total field, where the Sommerfeld transformant f has been determined via the procedure described above. Herein we are interested in the expressions for the excitation coefficients of the reflected and transmitted surface waves as well as in finding the diffraction coefficient of the circular wave in the far-field asymptotics. To this end, we deform the double-loop Sommerfeld contour γ in (16) into the steepest-descent paths $\gamma_\pm$ (Figure 4). In the process of such a deformation, the poles of the transformant f can be captured. The pole at $\alpha = \varphi_s := \Phi - \vartheta_+$ with unit residue gives rise to the incident surface wave U^i, see the expression for f_i after (33).

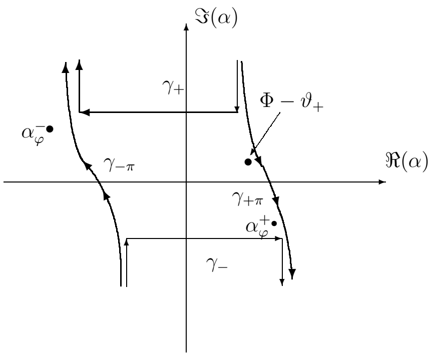

Figure 4. Deformation of the contour $\gamma=\gamma_+\cup\gamma_-$, $\alpha_\varphi^\pm=-\varphi\pm(\pi+\Phi+\vartheta_\pm)$.

We observe that this pole is captured in the process of deformation of the Sommerfeld contour γ into the SD paths $\gamma_{\pm\pi}$ provided $0\leqslant\Phi-\varphi<\pi-\mathrm{gd}(-\Im\theta_+)$,[8] i.e., for some angular neighborhood of l^1 in Ω. The pole will never be located in a close (of $O([kr]^{-1/2})$) neighborhood of the saddle point π because θ_+ is purely imaginary. However, if φ varies, the complex pole at $\alpha=\Phi-\vartheta_+-\varphi$ can be close to and even cross the SD path $\gamma_{+\pi}$. A uniform asymptotics is then represented in terms of a Fresnel-type integral with complex argument. By analogy with coalescence of the saddle point π and a real pole in the Malyuzhinets problem, the respective direction of observation $\varphi=\varphi_*$ can be called a singular direction. A similar phenomenon is obviously observed for the reflected and transmitted surface waves. We do not consider such "complex" singular directions in the leading approximation and ignore them in the nonuniform asymptotic expression of the far field and, therefore, in the numerical results.

The pole at $\alpha_*^+=\pi+\Phi+\vartheta_+$ belongs to the strip $\Pi(\Phi,3\Phi)$, and the residue is specified as follows.

Consider the first equation in (18) written in the form

$$f(\alpha)=-f(2\Phi-\alpha)\frac{\sin(\alpha-\Phi)-\sin\vartheta_+}{\sin(\alpha-\Phi)+\sin\vartheta_+},$$

then on the right-hand side the argument of $f(\,\cdot\,)$ is from the strip $\Pi(-\Phi,\Phi)$ if $\alpha\in\Pi(\Phi,3\Phi)$ and $f(2\Phi-\alpha)$ is regular there. However, the denominator of the fraction on the right-hand side is zero at $\alpha_*^+=\pi+\Phi+\vartheta_+$ and the respective

[8]The Gudermann function is defined as $\mathrm{gd}(x)=\mathrm{sign}(x)\arccos(1/\cosh x)$; see, for instance, [15].

residue of f at this pole is

$$C_s^+ = \mathrm{res}_{z=\alpha_*^+} f(z) = -2\tan\vartheta_+ f(\Phi - \pi - \vartheta_+) \tag{41}$$

with $\Phi - \pi - \vartheta_+ \in \Pi(-\Phi, \Phi)$. The excitation coefficient C_s^+ in (41) is specified by the value $f(\Phi-\pi-\vartheta_+)$, that is, it requires solution of the integral equations in order to determine f. The reflected surface wave propagating along l^1 takes the form

$$u_s^+(r,\varphi) = C_s^+ \mathrm{e}^{\mathrm{i}kr\cos[\varphi-\Phi-\vartheta_+]}$$

if $0 \leqslant \Phi - \varphi \leqslant -\mathrm{gd}(\Im(\vartheta_+))$, otherwise, the pole is not captured.

Taking into account the contribution of the saddle points $\pm\pi$, we arrive at the far-field asymptotics for the scattered circular wave, which is valid outside some close neighborhoods of l^1 and l^4,

$$u^{sc} \sim \frac{e^{\mathrm{i}kr+\mathrm{i}\pi/4}}{\sqrt{2\pi kr}} D_f(\varphi)\Big(1 + O\Big(\frac{1}{kr}\Big)\Big),$$

where $D_f(\varphi) = f(-\pi+\varphi) - f(\pi+\varphi)$, $|\varphi| \leqslant \Phi$. The expression of the far field is ($|\varphi| \leqslant \Phi$)

$$u(r,\varphi) = U^i(r,\varphi) + u_s^+(r,\varphi) + \frac{e^{\mathrm{i}kr+\mathrm{i}\pi/4}}{\sqrt{2\pi kr}} D_f(\varphi)\Big(1 + O\Big(\frac{1}{kr}\Big)\Big) \tag{42}$$

if $0 \leqslant \Phi - \varphi \leqslant \pi + \mathrm{gd}(\Im(\vartheta_+))$, otherwise, the first summand on the right-hand side of (42) is omitted. We observe that, in view of our assumptions, the poles cannot be close to the saddle points $\pm\pi$.

In the same manner, making use of the Sommerfeld representation for u in (17) and the second equation in (20), written as

$$h(\alpha) = -h(-2\Phi_e^- - \alpha)\frac{\sin(\alpha + \Phi_e^-) + \sin\vartheta_-}{\sin(\alpha + \Phi_e^-) - \sin\vartheta_-},$$

we compute the pole $\alpha_*^- = -(\pi + \Phi_e^- + \vartheta_-)$ and the transmitted surface wave propagating along l^4:

$$u_s^-(\rho_e,\varphi_e) = C_s^- \mathrm{e}^{\mathrm{i}k\rho_e\cos[\varphi_e+\Phi_e^-+\vartheta_-]}$$

if $0 \leqslant \Phi_e^- + \varphi_e \leqslant -\mathrm{gd}(\Im(\vartheta_-))$, otherwise, $u_s^- = 0$, with the excitation coefficient

$$C_s^- = 2\tan\vartheta_- h(\pi - \Phi_e^- + \vartheta_-).$$

Provided $\epsilon - \Phi_e^- \geqslant \varphi_e \geqslant -\Phi_e^-$, $\epsilon > 0$, i.e., in some angular neighborhood of l^4, we obtain the asymptotics

$$u(\rho_e,\varphi_e) = u_s^-(\rho_e,\varphi_e) + \frac{e^{\mathrm{i}k\rho_e+\mathrm{i}\pi/4}}{\sqrt{2\pi k\rho_e}} D_h(\varphi_e)\Big(1 + O\Big(\frac{1}{k\rho_e}\Big)\Big) \tag{43}$$

with $D_h(\varphi_e) = h(-\pi+\varphi_e) - h(\pi+\varphi_e)$.

In the overlapping domain, where both asymptotics (42) and (43) are valid, they give the same asymptotic values of the far field. The obtained asymptotic expressions enable us to assert that the radiation condition, postulated for the solution under study, is satisfied. Verification of Meixner's conditions at the vertices is performed for the Sommerfeld integral representation in a standard way (see [15, Chapter 1]).

§7. Numerical solution

To begin with, we recall that the Fredholm property discussed above in §5 ensures convergence and stability of the quadrature method applied to solving numerically the integral equation (40).

7.1. Numerical solution of the matrix integral equation. To compute the far-field asymptotics detailed in §6, in particular, the expressions (42) and (43), we need to solve the matrix integral equation (40) with its entries defined in equations (35)–(38). For the sake of efficiency, we exploit known properties of the integrands. As mentioned in footnote 5, all the involved integrands are even in the variable of integration τ, which halves the interval of integration. Furthermore, the asymptotic behavior of the Sommerfeld spectra (transformants) for an impedance wedge was given in (24).

On use of these properties we rewrite, as an example, the operator K_{21} as follows:

$$K_{21}V_f = \frac{\mathrm{i}}{2\pi}\int_0^{+\infty} I_{21}(\alpha, x)J_f(x)V_f(\tau(x))\exp(-x)\mathrm{d}x, \tag{44}$$

with $\tau(x) = \mathrm{i}x/\mu$ and

$$I_{21}(\alpha, x) = \frac{\psi_f(\alpha+\pi-\Phi)}{1+\mathrm{e}^{-2x}+2\mathrm{e}^{-x}\sin\mu(\alpha+\pi-\Phi)} + \frac{R_1^+(\alpha)\psi_f(-\alpha+\pi-\Phi)}{1+\mathrm{e}^{-2x}+2\mathrm{e}^{-x}\sin\mu(-\alpha+\pi-\Phi)},$$

$$J_f(x) = \frac{\exp\left(\mathrm{i}kd_1\cos\tau\right)\left(1-\mathrm{e}^{-2x}\right)}{\psi_f(\tau-\Phi)\mathrm{e}^{-x}}.$$

As suggested by equation (44), the Gauss–Laguerre scheme is an obvious choice for evaluating numerically the operator K_{21}, hence leading to

$$K_{21}V_f \approx \frac{\mathrm{i}}{2\pi}\sum_{m=1}^{L_f} w_m^{L_f} I_{21}(\alpha, x_m^{L_f})J_f(x_m^{L_f})V_f(\tau(x_m^{L_f})), \tag{45}$$

with $w_m^{L_f}$ and $x_m^{L_f}$ the weights and the abscissae of the Gauss–Laguerre scheme of order L_f.

To discretize the second line of (40), that is (36), we enforce the fulfilment of the above relation at certain points

$$\alpha = \mathrm{i}\, x_\ell^{L_{g^+}}/\mu_-,$$

where the $x_\ell^{L_{g^+}}$ are the abscissae of the Gauss–Laguerre scheme of order L_{g^+}, as required by the quadrature method. Thus we arrive at a linear system of algebraic equations that corresponds to the second line of the matrix integral equation (40):

$$V_{g^+}(\mathrm{i}\, x_\ell^{L_{g^+}}/\mu_-) = \frac{\mathrm{i}}{2\pi}\sum_{m=1}^{L_f} w_m^{L_f} I_{21}(\mathrm{i}\, x_\ell^{L_{g^+}}/\mu_-, x_m^{L_f}) J_f(x_m^{L_f}) V_f(\tau(x_m^{L_f})), \qquad \ell = 1, 2, \ldots, L_{g^+}. \tag{46}$$

Therefore, by applying the same procedure to the remaining ones of (40), we get a linear system of algebraic equations out of (40).

Rather than solving directly the discretized version of (40), we reduce its rank by making use of the second and the third line of the matrix integral equation (40), namely

$$V_g^+ = K_{21} V_f + V_g^{+,i}, \quad V_g^- = K_{34} V_h + V_g^{-,i} \tag{47}$$

in the remaining ones of (40). In this way we have a matrix integral equation with two unknown functions

$$\begin{pmatrix} V_f \\ V_h \end{pmatrix} = \begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{14} \\ \widetilde{K}_{41} & \widetilde{K}_{44} \end{pmatrix} \begin{pmatrix} V_f \\ V_h \end{pmatrix} + \begin{pmatrix} \widetilde{V}_f^i \\ \widetilde{V}_h^i \end{pmatrix}, \tag{48}$$

with

$$\begin{aligned} \widetilde{K}_{11} &= K_{12} K_{21}, \quad \widetilde{K}_{14} = K_{13} K_{34}, \\ \widetilde{K}_{41} &= K_{42} K_{21}, \quad \widetilde{K}_{44} = K_{43} K_{34}, \\ \widetilde{V}_f^i &= K_{12} V_g^{+,i} + K_{13} V_g^{-,i} + V_f^i, \\ \widetilde{V}_h^i &= K_{42} V_g^{+,i} + K_{43} V_g^{-,i} + V_h^i. \end{aligned}$$

7.2. Computation of the Sommerfeld transformants f, g, and h. The Sommerfeld transformants f, g, and h in their respective basic strips are obtained via an integral extrapolation of the values of the auxiliary transformants V_f, V_{g^+}, V_{g^-}, and V_h along the positive half of the imaginary axis in the complex τ-plane, with the aid of (34). For example, to calculate $f(\alpha)$, the first one of (34) is rewritten as:

$$f(\alpha) = \frac{\mathrm{i}\psi_f(\alpha)}{2\pi} \int\limits_0^{+\infty} I_f(\alpha, x) J_f(x) V_f(\tau(x)) \exp(-x)\, \mathrm{d}x, \tag{49}$$

with $I_f(\alpha, x) = 1/(1 + \mathrm{e}^{-2x} + 2\,\mathrm{e}^{-x} \sin \mu\alpha)$ and J_f already defined in Subsection 7.1. We evaluate (49) also with the Gauss–Laguerre scheme of order L_f, because this allows us to use the precalculated values of V_f and J_f at exactly the same abscissae.

To compute the scattering diagram $D_f(\varphi) = f(-\pi + \varphi) - f(\pi + \varphi)$ for $-\Phi \leqslant \varphi \leqslant \Phi$ with formula (42), we need to know the spectrum $f(\alpha)$ in the strip $\Pi(-\pi - \Phi, \pi + \Phi)$.

The expression for $f(\alpha)$ given in (34) is valid merely in the strip $\Pi(-\Phi, 3\Phi)$, hence it is necessary to extend $f(\alpha)$ to the left strip $\Pi(-\pi - \Phi, -\Phi)$.

This is carried out with the aid of the Malyuzhinets functional equation, to be more precise the second one of (18) rewritten as:

$$\begin{aligned} f(\alpha) = \frac{-\sin(\alpha + \Phi) - \sin\vartheta_1}{\sin(\alpha + \Phi) - \sin\vartheta_1} \Big[f(-\alpha - 2\Phi) - \mathrm{e}^{\mathrm{i}kd_1 \cos(\alpha+\Phi)} g(-\alpha - 2\Phi) \Big] \\ + \mathrm{e}^{\mathrm{i}kd_1 \cos(\alpha+\Phi)} g(\alpha). \end{aligned} \tag{50}$$

From $\alpha \in \Pi(-\pi - \Phi, -\Phi)$, it follows that $-\alpha - 2\Phi \in \Pi(-\Phi, \pi - \Phi)$. As is known, the expression for $g(\alpha)$ given in (34) holds good in the strip $\Pi(-\Phi_2^-, \pi - \Phi)$ and $\Phi_2^- > \Phi > \pi/2$, what remains to do is to extend $g(\alpha)$ to the left strip $\Pi(-\pi - \Phi, -\Phi_2^-)$.

To this end, we use the second one of (19), namely:

$$\begin{aligned} g(\alpha) = {} & \frac{-\sin(\alpha + \Phi_2^-) - \sin\vartheta_2}{\sin(\alpha + \Phi_2^-) - \sin\vartheta_2} \\ & \times \Big[g(-\alpha - 2\Phi_2^-) - \mathrm{e}^{\mathrm{i}kd_2 \cos(\alpha+\Phi_2^-)} h(-\alpha - 2\Phi_2^-) \Big] + \mathrm{e}^{\mathrm{i}kd_2 \cos(\alpha+\Phi_2^-)} h(\alpha). \end{aligned} \tag{51}$$

From $\alpha \in \Pi(-\pi - \Phi, -\Phi_2^-)$, it turns out that $-\alpha - 2\Phi_2^- \in \Pi(-\Phi_2^-, \pi + \Phi - 2\Phi_2^-)$. Furthermore, the expression for $h(\alpha)$ given in (34) is valid inside the strip $\Pi(-2\Phi_+ - \Phi_e^-, \pi - \Phi_2^-)$. This completes the required analytic continuation of $g(\alpha)$ to the strip $\Pi(-\pi - \Phi, -\Phi_2^-)$ and, hence, $f(\alpha)$ to the strip $\Pi(-\pi - \Phi, -\Phi)$.

As defined in (43), the scattering diagram reads $D_h(\varphi_e) = h(-\pi + \varphi_e) - h(\pi + \varphi_e)$ with $-\Phi_e^- \leqslant \varphi_e \leqslant \pi - \Phi_2^-$. This implies that we need $h(\alpha)$ in the strip $\Pi(-\pi - \Phi_e^-, \pi + \pi - \Phi_2^-)$. As already mentioned, $h(\alpha)$ from (34) is true for the strip $\Pi(-2\Phi_+ - \Phi_e^-, \pi - \Phi_2^-)$, making necessary its analytic continuation to the right, to be more precise, to the strip $\Pi(\pi - \Phi_2^-, 2\pi - \Phi_2^-)$.

We use the first equation of (20) rewritten below

$$h(\alpha) = \frac{-\sin(\alpha+\Phi_2^- - \pi) + \sin\vartheta_2}{\sin(\alpha+\Phi_2^- - \pi) + \sin\vartheta_2} \times \left[h(-\alpha - 2[\Phi_2^- - \pi]) - \mathrm{e}^{\mathrm{i}kd_2\cos(\alpha+\Phi_2^- - \pi)} g(-\alpha - 2[\Phi_2^- - \pi])\right] + \mathrm{e}^{\mathrm{i}kd_2\cos(\alpha+\Phi_2^- - \pi)} g(\alpha).$$

The requirement that α be in the strip $\Pi(\pi - \Phi_2^-, 2\pi - \Phi_2^-)$ leads to $-\alpha - 2[\Phi_2^- - \pi] \in \Pi(-\Phi_2^-, \pi - \Phi_2^-)$. As is known, $g(\alpha)$ given in (34) holds true in the strip $\Pi(-\Phi_2^-, \pi - \Phi)$. Hence the last step lies in extending $g(\alpha)$ to the right strip $\Pi(\pi - \Phi, 2\pi - \Phi_2^-)$, by means of the first one of (19)

$$g(\alpha) = \frac{-\sin(\alpha+\Phi-\pi)+\sin\vartheta_1}{\sin(\alpha+\Phi-\pi)+\sin\vartheta_1} \times \left[g(-\alpha-2[\Phi-\pi]) - \mathrm{e}^{ikd_1\cos(\alpha+\Phi-\pi)} f(-\alpha-2[\Phi-\pi])\right] + \mathrm{e}^{ikd_1\cos(\alpha+\Phi-\pi)} f(\alpha).$$

This can be verified by noting that for $\alpha \in \Pi(\pi - \Phi, 2\pi - \Phi_2^-)$ we have $-\alpha - 2[\Phi - \pi] \in \Pi(\Phi_2^- - 2\Phi, \pi - \Phi) \subset \Pi(-\Phi, \Phi)$.

7.3. An example. Figure 5 displays $|u^{sc}|$, the nonuniform scattered far field for diffraction of an incident surface wave by an impedance polygon, as given by formulas (42) (save for the incident surface wave $U^i(r,\varphi)$) and (43). The respective wedge angles are $\Phi = \Phi_- = \Phi_+ = 4\pi/7$, the complex angles of the surface impedances read $\vartheta_+ = -0.2711\mathrm{i}, \vartheta_1 = \vartheta_2 = 0.1 - 0.5\mathrm{i}$, and $\vartheta_- = -0.7\mathrm{i}$; the normalized distances between neighboring vertices are $kd_1 = 8\pi$ and $kd_2 = 4\pi$. For the numerical solution of the matrix integral equation (40), each of the auxiliary functions $V_f, V_{g^\pm}$, and V_h is disretized at the $L_f = 162$ abscissae. Details about the numerical solution like accuracy and rate of convergence will be reported elsewhere.

§8. Conclusion

In this work we have studied scattering of an incident surface wave in a polygonal domain with impedance boundary and derived expressions for the excitation coefficients of the reflected and transmitted surface waves as well as for the diffraction coefficient of the circular wave. The uniqueness of the classical solution has also been addressed.

The solution is constructed in the form of Sommerfeld integrals and by the use of a known extension of the Sommerfeld–Malyuzhinets (SM) technique reducing the original diffraction problem to a system of functional equations

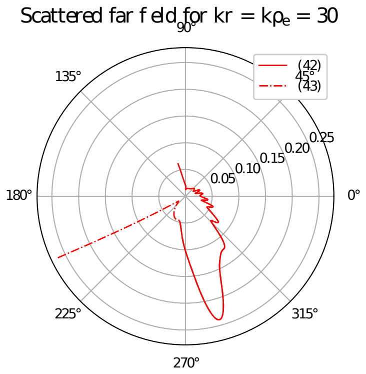

Figure 5. Scattered nonuniform far field $|u^{sc}|$ in diffraction of an incident surface wave by an impedance polygon (see Subsection 7.3 for details).

(Malyuzhinets equations). The respective Malyuzhinets equations are then reduced to a matrix Fredholm integral equation of the second kind and its solvability is then studied. Some appropriate estimates are obtained, which are necessary for the application of the so-called analytic Fredholm alternative. The integral equation depends on the wave number analytically and is uniquely solvable except for some discrete set of its characteristic values.

We have also discussed some geometrical restrictions on the angles of the polygonal boundary as well as those for the surface impedances. Numerical solution of the scattering problem is obtained and some physical analysis of the solution is also addressed.

§9. Appendix. Extension of the Sommerfeld–Malyuzhinets technique for the derivation of the Malyuzhinets equations

In this appendix we discuss an approach that enables us to derive the Malyuzhinets functional equations for the unknown Sommerfeld transformants f, g, h from the impedance boundary conditions. The results that we develop coincide with those in [18], however, we make use of a similar but, nevertheless, different way.

We exploit the representation for the total field (16) and, owing to the symmetry of the integration contour $\gamma = \gamma_+ \cup \gamma_-$, write it in the form

$$u(r, \varphi) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma_+} \mathrm{d}\alpha \, \mathrm{e}^{-\mathrm{i}kr\cos\alpha} [f(\alpha + \varphi) - f(-\alpha + \varphi)], \tag{52}$$

where φ is fixed, $|\varphi| \leqslant \Phi$. In what follows we assume that $|\arg(-\mathrm{i}k)| \leqslant \pi/2$ and vary k in this domain if necessary. We further assume that $|u(r,\varphi)| \leqslant C\, r^{a-1}$ for some $a > 0$ and $r \to 0$ as well as $|u(r,\varphi)| \leqslant C\, \exp\{br\}$, $b > 0$, $r \to \infty$. For a fixed φ, $u(\,\cdot\,,\varphi)$ is regular in the domain $|\arg(r)| < \epsilon_1$ for some $\epsilon_1 > 0$. We now deform γ_+ into a narrower loop γ_+^ϵ and apply the theorem by Malyuzhinets [22] on inversion of the Sommerfeld integral, thus obtaining

$$f(\alpha+\varphi) - f(-\alpha+\varphi) = \mathrm{i}k\sin\alpha \int\limits_{+0}^{+\infty} \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr\cos\alpha}\, u(r,\varphi). \tag{53}$$

By integrating by parts in the Sommerfeld integral, we also have the representation

$$\frac{1}{r}\frac{\partial u}{\partial\varphi} = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma_+} \mathrm{d}\alpha\, (-\mathrm{i}k\sin\alpha)\, \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\big[f(\alpha+\varphi) + f(-\alpha+\varphi)\big],$$

and obtain, from the inversion theorem,

$$f(\alpha+\varphi) + f(-\alpha+\varphi) = -\int\limits_{+0}^{+\infty} \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr\cos\alpha}\, \frac{1}{r}\frac{\partial u}{\partial\varphi}(r,\varphi). \tag{54}$$

From formulas (53), (54) we arrive at the desired integral relation connecting the Sommerfeld transformant f with the wave field u and its derivative $\frac{1}{r}\frac{\partial u}{\partial\varphi}$

$$f(\alpha+\varphi) = \frac{1}{2}\int\limits_0^\infty \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr\cos\alpha}\, \left\{\mathrm{i}k\sin\alpha\, u(r,\varphi) - \frac{1}{r}\frac{\partial u}{\partial\varphi}(r,\varphi)\right\}$$

or

$$f(\alpha) = \frac{1}{2}\int\limits_0^\infty \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr\cos(\alpha-\varphi)}\, \left\{\mathrm{i}k\sin(\alpha-\varphi)\, u(r,\varphi) - \frac{1}{r}\frac{\partial u}{\partial\varphi}(r,\varphi)\right\}. \tag{55}$$

In a next step we represent formula (55) in an invariant form. Introduce a ray $l = \{(r,\varphi)\colon r > 0,\ \varphi \text{ is fixed}\}$ and n is a vector normal to l, $\frac{1}{r}\frac{\partial u}{\partial\varphi}\big|_l = \frac{\partial u}{\partial n}\big|_l$, thus have

$$f(\alpha) = \frac{1}{2}\int\limits_l \mathrm{d}l(r,\varphi)\left\{\frac{\partial \mathrm{e}^{\mathrm{i}kr\cos(\alpha-\varphi)}}{\partial n}u(r,\varphi) - \mathrm{e}^{\mathrm{i}kr\cos(\alpha-\varphi)}\frac{\partial u}{\partial n}(r,\varphi)\right\} \tag{56}$$

and

$$f(\alpha) = \frac{1}{2}\int\limits_l \mathrm{d}l(r,\varphi)\left\{\frac{\partial E_\alpha(r,\varphi)}{\partial n}u(r,\varphi) - E_\alpha(r,\varphi)\frac{\partial u}{\partial n}(r,\varphi)\right\} \tag{57}$$

with $E_\alpha(r,\varphi) = \mathrm{e}^{\mathrm{i}kr\cos(\alpha-\varphi)}$.

The derivation of formulas (56), (57) is, in a sense, formal. Then we assume some limitations on the parameters α, φ, and k such that the integrals converge and specify analytic function with respect to α. We imply that u satisfies Meixner's condition near $r = 0$ and $|\varphi| \leqslant \Phi$. Moreover, $|\alpha|$ is sufficiently large and α is inside the loop γ_+^ϵ,

$$\Re\left(-\mathrm{i}kr\cos(\alpha-\varphi)\right) > 0 \tag{58}$$

and such that the integrals in (56), (57) converge. We intend to apply Green's theorem to the integral in (57) and deform the contour of integration l into $L_{0,\infty}$ to the right-hand side in Ω, which implies that the ends (i.e., 0 and ∞) of them coincide. Here $L_{0,\infty}$ is an arbitrary curve connecting $r = 0$ and $r = \infty$, for instance, along some curve $r = r(\varphi)$. To this end, we need an estimate on the arc $S_R(\varphi_1,\varphi_0) = \{(r,\varphi)\colon r = R, \quad \varphi \in (\varphi_1,\varphi_0)\}$ analogous to that in (58) for some φ_1, φ_0. Taking the asymptotics (9), (10) into account, we can consider $\alpha'' > 0$ sufficiently large ($\alpha = \alpha' + \mathrm{i}\alpha''$) and conclude that

$$\cos(\arg[-\mathrm{i}kR])\cos(\alpha'-\varphi)\cosh\alpha'' + \sin(\arg[-\mathrm{i}kR])\sin(\alpha'-\varphi)\sinh\alpha'' > 0 \tag{59}$$

is valid on the arc $S_R(\varphi_1,\varphi_0)$. The domain of the complex variable α, where inequality (59) is satisfied, is further denoted by $D_{[\arg(k),\varphi]}$. It parametrically depends on $\arg(k), \varphi$. For large $\alpha'' > 0$, from (59) one concludes that $\cos(\alpha' - \varphi - \arg[-\mathrm{i}kR])) > 0$ and

$$-\pi/2 + \varphi + \arg[-\mathrm{i}kR] < \alpha' < \pi/2 + \varphi + \arg[-\mathrm{i}kR].$$

We make use of the Green's theorem, noticing that $E_\alpha(r,\varphi)$ is a solution of the Helmholtz equation, and deform the integration contour l in (57) into $L_{0,\infty}$ so that

$$f(\alpha) = -\frac{1}{2}\int\limits_{L_{0,\infty}} \mathrm{d}l(r,\varphi)\ \left\{\frac{\partial E_\alpha(r,\varphi)}{\partial n}\,u(r,\varphi)\ -\ E_\alpha(r,\varphi)\,\frac{\partial u}{\partial n}(r,\varphi)\right\}, \tag{60}$$

where n is now the normal to $L_{0,\infty}$ directed into the exterior of the domain covered in the process of the deformation. In particular, we can take $L_{0,\infty} = l_{-\Phi} := \{(r,\varphi)\colon r > 0, \quad \varphi = -\Phi\}$, i.e., a ray, a part of which coincides with the segment A_1A_2, thus having

$$f(\alpha) = -\frac{1}{2}\int\limits_0^\infty \mathrm{d}r\mathrm{e}^{\mathrm{i}kr\cos(\alpha+\Phi)}\left\{-\mathrm{i}k\sin(\alpha+\Phi)u(r,-\Phi) + \frac{1}{r}\frac{\partial u}{\partial\varphi}(r,-\Phi)\right\}, \tag{61}$$

$\frac{1}{r}\frac{\partial}{\partial\varphi}\big|_{\varphi=-\Phi} = -\frac{\partial}{\partial n}\big|_{l_{-\Phi}}$. Remark that, instead of the ray in (61), one could take a polygonal line composed of segments.

A further important step is related to the representation of (61), splitting the contour, in the form

$$
\begin{aligned}
f(\alpha) = &-\frac{1}{2}\int_0^{d_1} \mathrm{d}r \mathrm{e}^{\mathrm{i}kr\cos(\alpha+\Phi)}\Big\{-\mathrm{i}k\sin(\alpha+\Phi)u(r,-\Phi)+\frac{1}{r}\frac{\partial u}{\partial\varphi}(r,-\Phi)\Big\} \\
&-\frac{1}{2}\int_{L_{d_1,\infty}} \mathrm{d}l(r,\varphi)\Big\{\frac{\partial E_\alpha(r,\varphi)}{\partial n}u(r,\varphi)-E_\alpha(r,\varphi)\frac{\partial u}{\partial n}(r,\varphi)\Big\},
\end{aligned}
\tag{62}
$$

where the second integral in (62) can be written in the form

$$
-\frac{1}{2}\int_{L_{d_1,\infty}} \mathrm{d}l(r,\varphi)\Big\{\frac{\partial E_\alpha(r,\varphi)}{\partial n}u(r,\varphi)-E_\alpha(r,\varphi)\frac{\partial u}{\partial n}(r,\varphi)\Big\} = \mathrm{e}^{\mathrm{i}kd_1\cos(\alpha+\Phi)}g(\alpha),
$$

$$
g(\alpha) = -\frac{1}{2}\int_{L_{0,\infty}} \mathrm{d}l(\rho_2,\varphi_2)\Big\{\frac{\partial E_\alpha(\rho_2,\varphi_2)}{\partial n}u(\rho_2,\varphi_2)-E_\alpha(\rho_2,\varphi_2)\frac{\partial u}{\partial n}(\rho_2,\varphi_2)\Big\}
$$

with $r=\rho_2+d_1$, $\varphi_2=-\Phi$. The last integral is also valid for any φ_2 such that $-\Phi_2^-\leqslant\varphi_2\leqslant\pi-\Phi$ and $\alpha\in D_{[\arg(k),\varphi_2]}$. As a result, g is the Sommerfeld transformant of the Sommerfeld integral representation of the wave field in the polar coordinates (ρ_2,φ_2) attributed to the point A_2

$$
u(\rho_2,\varphi_2) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}\alpha \mathrm{e}^{-\mathrm{i}k\rho_2\cos\alpha}g(\alpha+\varphi_2),
\tag{63}
$$

$$
g(\alpha) = -\frac{1}{2}\int_{L_{0,\infty}} \mathrm{d}\rho_2 \mathrm{e}^{\mathrm{i}k\rho_2\cos(\alpha-\varphi_2)}\Big\{-\mathrm{i}k\sin(\alpha-\varphi_2)u(\rho_2,\varphi_2)+\frac{1}{\rho_2}\frac{\partial u}{\partial\varphi_2}(\rho_2,\varphi_2)\Big\}
$$

for $-\Phi_2^-\leqslant\varphi_2\leqslant\pi-\Phi$. (It is useful to remind the reader that one could expect the possibility of meromorphic continuation with respect to α from $D_{[\arg(k),\varphi_2]}$ to the whole complex plane in the representation for g.)

Recalling relation (62), we can write it in the form

$$
\begin{aligned}
&f(\alpha)-\mathrm{e}^{\mathrm{i}kd_1\cos(\alpha+\Phi)}g(\alpha) \\
&\quad = -\frac{1}{2}\int_{L_{0,d_1}} \mathrm{d}l\Big\{-\mathrm{i}k\sin(\alpha+\Phi)\mathrm{e}^{\mathrm{i}kr\cos(\alpha+\Phi)}u(r,-\Phi) \\
&\qquad\qquad -\mathrm{e}^{\mathrm{i}kr\cos(\alpha+\Phi)}\frac{\partial u}{\partial n}(r,-\Phi)\Big\},
\end{aligned}
\tag{64}
$$

where the right-hand side in (64) is a holomorphic function for $\alpha\in D_{[\arg(k),\Phi]}$.

Exploiting the last formula and the boundary condition (3) with $j = 2$ on the right-hand side, we obtain

$$(\sin\alpha - \sin\vartheta_1)\big[f(\alpha - \Phi) - \mathrm{e}^{\mathrm{i}kd_1\cos\alpha} g(\alpha - \Phi)\big] = -\frac{1}{2}\int_0^{d_1} \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr\cos\alpha}\Big\{ - \mathrm{i}k\sin^2\alpha\, u(r, -\Phi) - \frac{\sin\vartheta_1}{r}\frac{\partial u}{\partial\varphi}(r, -\Phi)\Big\}. \tag{65}$$

We change the variable $\alpha \to -\alpha$ in (65), which preserves the right-hand side, then arrive at the second functional equation in (18) which, by analytic continuation with respect to α, is valid on the whole complex plane.

The other functional equations are deduced in a similar way with due modifications.

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