Alternative proof of upper bound for spanning trees in a graph
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ABSTRACT. We give a proof of upper bound of spanning trees in a graph in terms of vertex degrees using linear algebra techniques and generalize it to the case of multigraphs. Also we obtain for which graphs the inequality turns into equality.

1 Introduction

Let $G = (V, E)$ denote the undirected connected (multi)graph without loops and $\tau(G)$ denote number of spanning trees in G. In this paper we want to find upper bound of $\tau(G)$.

A common approach for counting spanning trees is *Kirchhoff's Matrix Tree Theorem* and its corollary that $\tau(G)$ can be expressed in terms of Laplacian eigenvalues. There are many various upper bounds for $\tau(G)$ in terms of number of vertices, number of edges and degrees of vertices. In [5] the following upper bound for number of spanning trees in a graph was proved by induction using stronger result about multigraphs:

Theorem 1.1. *Let G be a simple graph, $d_1 \leqslant \dots \leqslant d_n$ — degrees of its vertices. Then*

$$\tau(G) \leqslant \frac{(1+d_1)\dots(1+d_n)}{n^2}.$$

This paper is organised as follows. We give an alternative proof of this inequality using linear algebraic techniques. After that we formulate the analogous result for multigraphs and prove its sharpness for complete multigraphs.

2 Preliminary lemmas

Def. Let $x = (x_1 \geqslant \dots \geqslant x_n)$ and $y = (y_1 \geqslant \dots \geqslant y_n)$ be two finite seqences of real numbers. We say that x **majorizes** y ($x \succ y$) iff for every $k \in [1..n]$

$$x_1 + \dots + x_k \geqslant y_1 + \dots + y_k$$

and $x_1 + \dots + x_n = y_1 + \dots + y_n$.

Lemma 2.1 (Karamata's inequality, [2])**.** *Let $f\colon I \to \mathbb{R}$ be a real-valued convex function, I is an interval on the real line, $x = (x_1 \geqslant \dots \geqslant x_n)$ and $y = (y_1 \geqslant \dots \geqslant y_n)$ are two finite sequences of numbers in I that $x \succ y$. Then the following inequality is true:*

$$f(x_1) + \dots + f(x_n) \geqslant f(y_1) + \dots + f(y_n).$$

The equality holds iff the sequences coincide, i.e. $x_i = y_i$ for every $i \in [1..n]$.

Corollary. If $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ are two sequences of positive numbers and $x \succ y$, then $x_1 \cdot \ldots \cdot x_n \leqslant y_1 \cdot \ldots \cdot y_n$.

Proof. Function $f(x) = -\log x$ is defined on $\mathbb{R}_+$ and it is convex. Applying Karamata's inequality to this function and given sequences we obtain

$$-\log x_1 - \dots - \log x_n \geqslant -\log y_1 - \dots - \log y_n.$$

Reversing the sign and exponentiating with base e, we obtain the desired inequality. □

Lemma 2.2 (Schur's inequality[4])**.** *Let A be the symmetric real-valued $n \times n$ matrix with diagonal elements $d_1 \geqslant \dots \geqslant d_n$ and eigenvalues $\lambda_1 \geqslant \dots \geqslant \lambda_n$. Then $d_1 + \dots + d_k \leqslant \lambda_1 + \dots + \lambda_k$ for every $k \in [1..n]$.*

3 Basic properties about Laplacian eigenvalues

Def. Let G be a simple graph (without multiple edges and loops). **Laplacian** $L(G)$ of this graph is the following matrix:

$$L_{i,j} = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_iv_j \in E(G) \\ 0 & \text{if } i \neq j \text{ and } v_iv_j \notin E(G) \end{cases}$$

Proposition 1. *Let $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$ be the spectrum of $L(G)$. Then for every $k \in [1..n]$ $\mu_k \in [0, n]$.*

Proposition 2. *Let $0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$ be the spectrum of $L(G)$. Define **the complement** of G: for every $v_i, v_j \in V(G)$:*

$$v_iv_j \in E(G) \iff v_iv_j \notin E(\overline{G});$$
$$v_iv_j \notin E(G) \iff v_iv_j \in E(\overline{G}).$$

Then the spectrum of $L(\overline{G})$ is $0 \leqslant n - \mu_n \leqslant \cdots \leqslant n - \mu_2$.

Theorem 3.1 (Kirchhoff's theorem, [1])**.** *Number of spanning trees in graph G (denote by $\tau(G)$) is equal to every cofactor of $L(G)$.*

Corollary ([3])**.** If $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ is the spectrum of $L(G)$, then $\tau(G) = \frac{\mu_2 \cdot \ldots \cdot \mu_n}{n}$.

Now proceed to give the proof of main theorem.

4 Proof of the main theorem

Let us remind the main theorem.

Theorem 4.1. *Let G be a simple graph, $v(G) = n$, $d_1 \leqslant \cdots \leqslant d_n$ be its degree sequence. Then the following equality holds:*

$$\tau(G) \leqslant \frac{(1 + d_1) \cdot \ldots \cdot (1 + d_n)}{n^2}.$$

Proof. From corollary of Kirchhoff's theorem we know that if $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ are eigenvalues of $L(G)$, then $\tau(G) = \frac{\mu_2 \cdot \ldots \cdot \mu_n}{n}$. Using this equality, we obtain

$$\frac{\mu_2 \cdot \ldots \cdot \mu_n}{n} \leqslant \frac{(1 + d_1) \ldots (1 + d_n)}{n^2}.$$

Hence, we should prove the following inequality:

$$n \cdot \mu_n \cdot \ldots \cdot \mu_2 \leqslant (1 + d_n) \cdot \ldots \cdot (1 + d_1).$$

Notice that sums of multipliers in both parts are equal. Indeed,

$$n + \mu_n + \cdots + \mu_2 = n + \mathrm{Tr}(L(G)) = n + d_1 + \cdots + d_n = (d_n + 1) + \cdots + (d_1 + 1).$$

Now consider graph $\overline{G}$ — the complement of G. Its degree sequence is $n - 1 - d_n \leqslant \cdots \leqslant n - 1 - d_1$ and its spectrum is $0 \leqslant n - \mu_n \leqslant \cdots \leqslant n - \mu_2$. Applying Schur's inequality we obtain

$$(n - 1 - d_1) + \cdots + (n - 1 - d_{k-1}) \leqslant (n - \mu_2) + \cdots + (n - \mu_k),$$

which is equivalent to

$$\mu_2 + \cdots + \mu_k \leqslant d_1 + \cdots + d_{k-1} + (k - 1) = (1 + d_1) + \cdots + (1 + d_{k-1}).$$

So,

$$n + \mu_n + \cdots + \mu_{k+1} \geqslant (1 + d_n) + \cdots + (1 + d_k)$$

Thus, we get that sequence $(n, \mu_n, \ldots, \mu_2)$ majorizes $(1 + d_n, \ldots, 1 + d_1)$. Now apply the corollary of Karamata's inequality for products and get the desired inequality. ∎

4.1 Generalization for multigraphs

As a corollary, we give the variant of the inequality of the main theorem in case of multigraphs.

Def. Define the **complement** of multigraph G with maximal edge multiplicity Δ ($\Delta = \max_{i\neq j} |L_{i,j}(G)|$): for every edge $e \in E(G)$

$$\mu_G(e) = k \iff \mu_{\overline{G}}(e) = \Delta - k,$$

where $\mu_G(e)$ is multiplicity of e in G.

Proposition 3. *Let G be a multigraph with maximal edge multiplicity Δ, $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ — its spectrum. Then for every $k \in [1..n]$ $\mu_n \in [0, n\Delta]$.*

Proposition 4. *Let G be a multigraph with maximal edge multiplicity Δ, $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ — its spectrum. Then the spectrum of $L(\overline{G})$ is $0 \leqslant n\Delta - \mu_n \leqslant \cdots \leqslant n\Delta - \mu_2$.*

Theorem 4.2. *Let G be a multigraph, $v(G) = n$, Δ — its maximal edge multiplicity and $d_1 \leqslant \cdots \leqslant d_n$ — its degree sequence. Then the following inequality is true:*

$$\tau(G) \leqslant \frac{(\Delta + d_1)\dots(\Delta + d_n)}{\Delta n^2}$$

Proof. We will apply corollary of Karamata's inequality for sequences $(n\Delta, \mu_n, \dots, \mu_2)$ and $(\Delta + d_n, \dots, \Delta + d_1)$ in the similar fashion as in proof of the main theorem. Considering the multigraph $\overline{G}$ and using proposition about its eigenvalues, we obtain the following inequality for every $k \in [1..n]$:

$$((n-1)\Delta - d_1) + \cdots + ((n-1)\Delta - d_{k-1}) \leqslant (n\Delta - \mu_2) + \cdots + (n\Delta - \mu_k),$$

which is equivalent to

$$\mu_2 + \cdots + \mu_k \leqslant (d_1 + \Delta) + \dots (d_{k-1} + \Delta).$$

So,

$$n\Delta + \mu_n + \cdots + \mu_{k+1} \geqslant (d_n + \Delta) + \cdots + (d_k + \Delta).$$

We get that $(n\Delta, \mu_n, \dots, \mu_2) \succ (d_n + \Delta, \dots, d_1 + \Delta)$, as we needed. Application of Karamata's inequality for products completes the proof. □

Now consider in what graphs the inequality turns into equality.

Statement 1. *The inequality turns into equality iff G is a complete multigraph with all edge multiplicities equal to Δ.*

Proof. We have that $n\Delta \cdot \mu_n \cdot \dots \cdot \mu_2 = (d_n + \Delta) \dots (d_1 + \Delta)$. It is true iff the sequences coincide, i.e. $n\Delta = d_n + \Delta$, $\mu_n = d_{n-1} + \Delta$, $\dots$, $\mu_2 = d_1 + \Delta$. From the first equality we obtain that $d_n = (n-1)\Delta$, and since every edge of G has multiplicity no greater than Δ, we get that every edge incident to the vertex with degree d_n has multiplicity Δ. Notice that $\overline{G}$ has at least two connectivity components, thus its second eigenvalue is zero, $\mu_n = n\Delta$ and $d_{n-1} = (n-1)\Delta$. So, every edge incident to the vertex with degree d_{n-1} has multiplicity Δ. Iterating this process, we obtain that all non-zero Laplacian eigenvalues of G are equal to $n\Delta$ and every vertex has degree $(n-1)\Delta$. Thus, every edge has multiplicity Δ. □

ACKNOWLEDGEMENTS

The author is grateful to his advisor Fedor Petrov for suggestions and discussions concerned with this paper and his support. The main results are formulated in author's bachelor thesis.

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