ISSN 1063-4541, Vestnik St. Petersburg University. Mathematics, 2016, Vol. 49, No. 1, pp. 81–84. © Allerton Press, Inc., 2016.
Original Russian Text © S.A. Zegzhda, N.V. Naumova, Sh.Kh. Soltakhanov, M.P. Yushkov, 2016, published in Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1. Matematika, Mekhanika, Astronomiya, 2016, No. 1, pp. 000–000.

MECHANICS

Relationship Between the Udwadia–Kalaba Equations and the Generalized Lagrange and Maggi Equations

S. A. Zegzhda†[a], N. V. Naumova[a], Sh. Kh. Soltakhanov[b], and M. P. Yushkov[a]

[a] *St. Petersburg State University, Universitetskaya nab. 7–9, St. Petersburg, 199034 Russia*

[b] *Chechen State University, ul. A. Sheripova, 32, Grozny, 364051 Russia*

e-mail: zegzhdasa@mail.ru, nat_n75@mail.ru, yushkovmp@mail.ru, soltakhanov@yandex.ru

Abstract—In their paper “A New Perspective on Constrained Motion,” F. E. Udwadia and R. E. Kalaba propose a new form of matrix equations of motion for nonholonomic systems subject to linear nonholonomic second-order constraints. These equations contain all of the generalized coordinates of the mechanical system in question and, at the same time, they do not involve the forces of constraint. The equations under study have been shown to follow naturally from the generalized Lagrange and Maggi equations; they can be also obtained using the contravariant form of the motion equations of a mechanical system subjected to nonholonomic linear constraints of second order. It has been noted that a similar method of eliminating the forces of constraint from differential equations is usually useful for practical purposes in the study of motion of mechanical systems subjected to holonomic or classical nonholonomic constraints of first order. As a result, one obtains motion equations that involve only generalized coordinates of a mechanical system, which corresponds to the equations in the Udwadia–Kalaba form.

Key words: nonholonomic mechanics, linear nonholonomic second-order constraints, Udwadia–Kalaba equations, generalized second-order Lagrange equations with multipliers, generalized Maggi equations.

DOI: 10.3103/S1063454116010143

1. INTRODUCTION

In [1], F. E. Udwadia and R. E. Kalaba proposed a new form of matrix equations of motion for nonholonomic systems subject to second-order linear nonholonomic constraints. The number of equations corresponds to the number of generalized coordinates of the mechanical system under consideration, and the equations are not supposed to involve the forces of constraint. This is a great advantage of these equations, and hence the authors of the paper [1] note that “the equations of motion obtained in this paper appear to be the simplest and most comprehensive discovered so far.” These equations were derived using the fairly specific Moor–Penrose transform [2, 3]; this construction is referred in the Russian literature as the pseudo-inverse matrix. In the present paper, we show that the equations derived in [1] can be obtained in a natural way by employing the generalized Maggi and Lagrange equations or when using the contravariant form of motion equations of a mechanical system subject to second-order nonholonomic linear constraints.

2. THE USE OF THE TANGENT SPACE FOR REPRESENTATION IN THE VECTOR FORM OF THE MOTION OF NONHOLONOMIC SYSTEMS

We consider [4] the motion of a mechanical system with three degrees of freedom in the tangent space to the manifold of all its possible positions at a given time. The motion will be described in the curvilinear coordinate system $q = (q^1, ..., q^s)$, which has the principal and reciprocal bases $\{\mathbf{e}_\sigma\}$, $\{\mathbf{e}^\tau\}$, $\sigma, \tau = \overline{1, s}$, where

$$\mathbf{e}_\sigma \cdot \mathbf{e}^\tau = \delta^\tau_\alpha, \quad \sigma, \tau = \overline{1, s}, \tag{2.1}$$

and δ^τ_α is the Kronecker delta.

†Deceased.

Assume that the motion of this mechanical system is subject to constraints (precisely such constraints were considered in the paper [1]) as follows:

$$f_2^{\kappa}(t, q, \dot{q}, \ddot{q}) \equiv a_{\kappa}^{l+\kappa}(t, q, \dot{q})\ddot{q}^{\sigma} + a_0^{l+\kappa}(t, q, \dot{q}) = 0, \quad \kappa = \overline{1, k}, \quad l = s-k. \tag{2.2}$$

The first-order differentiated nonholonomic constraints $f_1^{\kappa}(t, q, \dot{q}) = 0$ and the holonomic constraints $f_1^{\kappa}(t, q) = 0$ ($\kappa = \overline{1, k}$) can be written in this form after differentiating them twice with respect to time. Furthermore, in form (2.2), one may also directly specify the second-order linear nonholonomic constraints (see, e.g., [5]). The first example of a program second-order nonlinear nonholonomic constraint realized in one problem in cosmonautics was also offered in [6].

For the study of motion, we introduce the following transformations between the generalized accelerations $\ddot{q} = (\ddot{q}, \ldots, \ddot{q}^s)$ and the pseudo-accelerations $w_* = (w_*^1, \ldots, w_*^s)$:

$$w_*^{\rho} = w_*^{\rho}(t, q, \dot{q}, \ddot{q}), \quad \ddot{q}^{\sigma} = \ddot{q}^{\sigma}(t, q, \dot{q}, w_*), \quad \rho, \sigma = \overline{1, s}. \tag{2.3}$$

Here, the variables t, q, $\dot{q}$ play the role of parameters. It is convenient to particularize the first group of transformations as follows:

$$w_*^{\lambda} = f_*^{\lambda}(t, q, \dot{q}, \ddot{q}), \quad \lambda = \overline{1, l} \quad (l = s-k), \quad w_*^{l+\kappa} = f_2^{\kappa}(t, q, \dot{q}, \ddot{q}), \quad \kappa = \overline{1, k}. \tag{2.4}$$

Due to the choice of quasi-velocities in form (2.4), they become zeros, while the first ones depend on the choice of the functions $f_*^{\lambda}(t, q, \dot{q}, \ddot{q})$, $\lambda = \overline{1.l}$.

By setting the direct and inverse transformations (2.3), (2.4), we split the s-dimensional space under consideration into the direct sum of the subspaces K and L with bases $\varepsilon^{l+\kappa} = (\partial w_*^{l+\kappa}/\partial \ddot{q}^{\tau})\mathbf{e}^{\tau} = a_{\sigma}^{l+\kappa}(t, q, \dot{q})\mathbf{e}^{\tau} \equiv \nabla'' f_2^{\kappa}$, $\kappa = \overline{1, k}$, respectively [4]. The generalized Hamiltonians used here $\boldsymbol{\varepsilon}_{\lambda} = (\partial \ddot{q}^{\sigma}/\partial w_*^{\lambda})\mathbf{e}_{\sigma}$, $\kappa = \overline{1, k}$, were introduced by Polyakhov [7]. Now, in the case of ideal constraints (2.2), the vector equation of motion of a mechanical system in the tangent space can be written in the following form (see [4])

$$M\mathbf{W} = \mathbf{Y} + \Lambda_{\kappa}\varepsilon^{l+\kappa}, \quad \kappa = \overline{1, k}. \tag{2.5}$$

Here, M is the mass of the entire mechanical system, $\mathbf{W}$ is the acceleration vector of the system, $\mathbf{Y}$ is the vector of forces acting on the system, and Λ_{κ} are the generalized forces of constraint.

3. GENERALIZED LAGRANGE AND MAGGI EQUATIONS

Multiplying equation (2.5) by the vectors $\mathbf{e}_{\rho}$, $\rho = \overline{1, s}$, and taking into account (2.1), we get the generalized second-order Lagrange equations with multipliers

$$MW_{\rho} \equiv \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\rho}} - \frac{\partial T}{\partial q^{\rho}} \equiv M(g_{\rho\sigma}\ddot{q}^{\sigma} + \Gamma_{\rho,\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}) = Q_{\rho} + \Lambda_{\kappa}\frac{\partial f_2^{\kappa}}{\partial \ddot{q}^{\rho}},$$
$$\rho, \sigma = \overline{1, s}, \quad \kappa = \overline{1, k}, \quad q^0 = t. \tag{3.1}$$

Here, the coefficients $g_{\rho\sigma}$ (and the analogous ones in the nonstationary basis, when $q^0 = t$) are given in terms of the kinetic energy, while the coefficients $\Gamma_{\rho,\alpha\beta}$ are the Christoffel symbols of the first kind and similar expressions in the nonstationary basis.

Next, if Eq. (2.5) is multiplied by the vectors ε_{ρ}, $\rho = \overline{1, s}$, then we obtain two groups of generalized Maggi equations

$$(MW_{\sigma} - Q_{\sigma})\frac{\partial \ddot{q}^{\sigma}}{\partial w_*^{\lambda}} = 0, \quad \lambda = \overline{1, l}, \quad \sigma = \overline{1, s}, \tag{3.2}$$

$$(MW_{\sigma} - Q_{\sigma})\frac{\partial \ddot{q}^{\sigma}}{\partial w_*^{l+\kappa}} = \Lambda_{\kappa}, \quad \kappa = \overline{1, k}, \quad \sigma = \overline{1, s}. \tag{3.3}$$

The so-obtained second-order Lagrange equations with multipliers and the Maggi equations are called generalized because they extend the corresponding equations of nonholonomic mechanics under first-order nonholonomic constraints to the case of constraints (2.2).

The contravariant form of motion equations is obtained by multiplying law (2.5) by the vectors $\mathbf{e}^\sigma$, $\sigma = \overline{1, s}$ as follows:

$$MW^\sigma \equiv M(\ddot{q}^\sigma + \Gamma^\sigma_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta) = Q^\sigma + \Lambda_\kappa g^{\sigma\rho}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho}, \quad \sigma, \rho = \overline{1, s}, \quad \kappa = \overline{1, k}.$$

Solving this system for the generalized accelerations, this establishes the following:

$$\ddot{q}^\sigma = Q^\sigma/M + \Lambda_\kappa g^{\sigma\rho}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho}/M - \Gamma^\sigma_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta, \quad \sigma, \rho = \overline{1, s}, \quad \kappa = \overline{1, k}. \tag{3.4}$$

Here, $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols of the second kind and the analogous functions and $g^{\sigma\rho}$ are the entries of the additional metric tensor. It is worth noting that this form of the contravariant equations is equivalent to the system of generalized second-order Lagrange equations with multipliers solved with respect to the generalized accelerations.

4. RELATION BETWEEN THE UDWADIA–KALABA EQUATIONS AND THE GENERALIZED LAGRANGE AND MAGGI EQUATIONS

The above differential equations (3.1), (3.2), and (3.4) with given initial data need to be integrated jointly with equations of constraint (2.2). Furthermore, the system of differential equations (3.2), (2.2) is written with respect to all unknown functions $q = (q^1, \ldots, q^s)$ and does not contain the Lagrange multipliers Λ_κ, $\kappa = \overline{1, k}$. In this respect, it proves to be of equal value with the Udwadia–Kalaba equations.

It is more difficult to solve the system of equations (3.1), (2.2) because, along with the derivatives of the generalized coordinates, it also contains the first powers of the unknown functions Λ_κ, $\kappa = \overline{1, k}$. We exclude the latter equations and compose the system of s differential equations with respect to the functions $q = (q^1, \ldots, q^s)$. Then, as was pointed out above, we obtain formulas (3.4). Substituting these expressions for the generalized accelerations in the equations of constraint, we obtain the system of algebraic equations with respect to the Lagrange multipliers

$$A^{\kappa\kappa^*}\Lambda_{\kappa^*} = B^\kappa, \quad \kappa, \kappa^* = \overline{1, k}, \tag{4.1}$$

where we set

$$A^{\kappa\kappa^*}(t, q, \dot{q}) = a_\sigma^{l+\kappa} g^{\sigma\rho}\frac{\partial f_2^{\kappa^*}}{\partial \ddot{q}^\rho}, \quad B^\kappa = a_\sigma^{l+\kappa}\Gamma^\sigma_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - a_\sigma^{l+\kappa}Q^\sigma - Ma_0^{l+\kappa}.$$

Solving system (4.1), we find the generalized forces of constraint (2.2) as functions of t, q, $\dot{q}$ as follows:

$$\Lambda_{\kappa^*} = C_{\kappa^*}(t, q, \dot{q}), \quad \kappa^* = \overline{1, k}, \quad C_{\kappa^*} = A_{\kappa^*\kappa}B^{;\kappa}.$$

Here, $A_{\kappa^*\kappa}$ are entries of the matrix inverse of the matrix $(A^{\kappa^*\kappa})$. Substituting the so-obtained Lagrange multipliers in formulas (4.1), we obtain the following required form of the differential equations of motion of the mechanical system:

$$\ddot{q}^\sigma = D^\sigma(t, q, \dot{q}), \quad \sigma = \overline{1, s}, \quad D^\sigma(t, q, \dot{q}) = Q^\sigma/M + C_{\kappa^*}g^{\sigma\rho}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho}/M - \Gamma^\sigma_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta. \tag{4.2}$$

It should be noted that a similar method of eliminating the constraint forces from differential equations is usually employed in practical purposes for studying the motion of mechanical systems subjected to holonomic or classical nonholonomic constraints of the first order.

There is another way to eliminate the Lagrange multipliers from equations (3.1). Substituting their expressions form (3.3) into equations (3.1), we have

$$E_{\rho\sigma}(t, q, \dot{q})\ddot{q}^\sigma = F_\rho(t, q, \dot{q}),$$

$$E_{\rho\sigma} = M\left(g_{\rho\sigma} - g_{\rho^*\sigma}\frac{\partial \ddot{q}^{\rho^*}}{\partial w_*^{l+\kappa}}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho}\right),$$

$$F_\rho = Q_\rho - Q_{\rho^*}\frac{\partial \ddot{q}^{\rho^*}}{\partial w_*^{l+\kappa}}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho} + M\Gamma_{\rho^*,\alpha\beta}\dot{q}^\alpha\dot{q}^\beta\frac{\partial \ddot{q}^{\rho^*}}{\partial w_*^{l+\kappa}}\frac{\partial f_2^\kappa}{\partial \ddot{q}^\rho} + -M\Gamma_{\rho,\alpha\beta}\dot{q}^\alpha\dot{q}^\beta, \quad \sigma, \rho, \rho^* = \overline{1, s}, \quad \kappa = \overline{1, k}.$$

As a result, we have

$$\ddot{q}^\sigma = G^\sigma(t, q, \dot{q}), \quad G^\sigma(t, q, \dot{q}) = E^{\sigma\rho}(t, q, \dot{q})F_\rho(t, q, \dot{q}), \quad \sigma, \rho = \overline{1, s}, \tag{4.3}$$

where $E^{\sigma\rho}$ are entries of the matrix inverse of the matrix $(E_{\rho\sigma})$.

Equations (4.2), (4.3), as obtained from the generalized second-order Lagrange equations with multipliers and the generalized Maggi equations, are the Udwadia–Kalaba equations in the tensor form. It is worth noting that these equations can also be obtained using the linear transformation of forces introduced in book [4].

REFERENCES

1. F. E. Udwadia and R. E. Kalaba, "A new perspective on constrained motion," Proc. R. Soc. London, Ser. A **439**, 407–410 (London, 1992).
2. E. H. Moore, "On the reciprocal of the general algebraic matrix," Bull. Am. Math. Soc. **26**, 394–395 (1920).
3. R. Penrose, "A generalized inverse of matrices," Math. Proc. Cambridge Philos. Soc. **51**, 406–413 (1955).
4. S. A. Zegzhda, Sh. Kh. Soltakhanov, and M. P. Yushkov, *Nonholonomic Mechanics. Theory and Applications* (Nauka, Fizmatlit, Moscow, 2009) [in Russian].
5. F. Kitzka, "An example for the application of a nonholonomic constraint of 2nd order in particle mechanics," Z. Angew. Math. Mech. **66**, 312–314 (1986).
6. Sh. Kh. Soltakhanov and M. P. Yushkov, "Equations of motion for a certain nonholonomic system in the presence of a 2nd order constraint," Vestn. Leningr. Univ., Ser. 1, No. 4, 26–29 (1991).
7. N. N. Polyakhov, "On differential principles in mechanics obtained from the equations of motion for nonholonomic systems," Vestn. Leningr. Univ., Ser. 1, No. 3, 106–116 (1974).

Translated by A. Alimov

SPELL:OK